Double Preconditioning for Gabor Frame Operators: Algebraic, Functional Analytic and Numerical Aspects

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Abstract

This paper provides algebraic and analytic, as well as numerical arguments why and how double preconditioning of the Gabor frame operator yields an efficient method to compute approximate dual (respectively tight) Gabor atoms for a given time-frequency lattice. We extend the definition of the approach to the continuous setting, making use of the so-called Banach Gelfand Triple, based on the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and show the continuous dependency of the double preconditioning operators on their parameters. The generalization allows to investigate the influence of the order of the two main single preconditioners (diagonal and convolutional). In the applied section we demonstrate the quality of double preconditioning over all possible lattices and adapt the method to approximate the canonical tight Gabor window, which yields a significant generalization of the FAB-method used in OFDM-applications. Finally, we demonstrate that our approach provides a way to efficiently compute approximate dual families for Gabor families which arise from a slowly varying pattern instead of a regular lattice.

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1. Introduction

Time-frequency analysis (TFA) constitutes one of the corner-stones of modern signal processing and is at the same time an essential part of the area of mathematics called harmonic analysis. What the fields of engineering and mathematics thereby connects, is the use of the Short-Time Fourier Transform (STFT), one of the central objects of both areas. Gabor systems arise here in a natural way as they are used for atomic decomposition of signals, respectively for the reconstruction of functions (or distribution) from a sampled STFT. Compared to wavelet theory the TFA setting has the advantage that it can be applied naturally in the context of finite Abelian groups, i.e. one can formulate the underlying theory for the finite-dimensional case in a mathematically rigorous way, providing a solid basis for implementations. In this sense, having the right tools for the efficient handling of Gabor-related operators is crucial to understand both, the finite discrete and the continuous setting. This is the main motivation for our work.

Generally speaking, preconditioning is an important approach in the numerical treatment of operators, because it allows to improve the computational properties of the given problem. Instead of solving $Ox = y$, one solves $(PO)\, x = Py$, where P can be efficiently calculated and PO has "nicer" numerical properties, for example, a better condition number. A typical approach is done as starting point for the Jacobi algorithm, where the inverse of the diagonal of a matrix is used as a preconditioning matrix, called *diagonal preconditioning*. This preconditioner is the first main ingredient in this paper.

Going one step further, one can also solve $(UOU^*)\, Ux = Uy$ for a unitary U and gets a different preconditioning operator. This was done in [4] by combining diagonal preconditioning with the Fourier Transform $\mathcal{F}$ as the unitary transformation to introduce a *circulant preconditioning* method. The implementation of the Fourier transform by the FFT is very efficient, so that such an approach is fruitful for improving numerical efficiency. This is the second main ingredient.

The consecutive application of the diagonal and (then) the circulant preconditioning approach was introduced in the same paper as the so-called *double preconditioning* method to compute *approximate dual Gabor atoms* at very low computational costs, which can be used to compute the true canonical dual (up to precision) in an iterative approach. Numerical experiments demonstrate that this approach often works impressively well.

In the present manuscript, we revisit this approach by providing further numerical insights and - in particular - working on the related theory, settling algebraic and functional analytic background questions which, in turn, paves the way for further applications and generalizations.

Before we start off, let us motivate the intentions of this paper by collecting several basic facts concerning *regular* Gabor frames, i.e. Gabor frames which are generated by applying time-frequency shifts from a lattice Λ in phase space to some Gabor atom g.

1.1. Gabor Motivation

It is a widely accepted viewpoint that the determination of a *dual frame* for a given Gabor frame is a computationally expensive task. A lot of research has been investigated in this aspect [49, 52] and even though computer power (and parallelization schemes) has evolved tremendously, it is still a hard task for multi-dimensional signals or high sampling rates. Given a lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and some Gabor atom g, a dual is needed for two main tasks: Either to reconstruct a signal from the samples of the STFT with window g at the lattice points - the frame theoretic aspect. Or, using the elements of the Gabor family as building blocks, to represent a given signal as a Gabor series (the atomic decomposition viewpoint). Both tasks require, for good numerical properties, a modest amount of *redundancy*, meaning a certain quasi-geometric density of Λ in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. We denote this by $\mathrm{red}(\Lambda)$. For some applications, it is also quite useful to have *tight Gabor frames* (with low redundancy) because such a tight system has an important conceptual bonus, as one can use the same atoms for analysis and synthesis, while still achieving perfect reconstruction (up to a constant). So, one does not have to distinguish between the system used in the atomic decomposition or for the samples of the STFT. For example, when cutting out signal parts in the time-frequency domain, e.g. by multipliers, the interpretation of the results is much more straightforward.

Although the regular case - i.e. the existence of a lattice - allows us to make use of a variety of structural properties of the corresponding frame operator, it is still of interest to have alternative methods, especially in preparation for the multi-dimensional (including non-separable cases, i.e. lattices which are not of the form $\Lambda = \Lambda_1 \times \Lambda_2$ with $\Lambda_1 \lhd G$ and $\Lambda_2 \lhd \widehat{G}$) or for a so-called *quasi-regular* situation, which behaves locally similarly to a lattice, but which do not allow to benefit from the algebraic structure arising in the

case of sets Λ which have the structure (such as the Janssen representation of $\mathbf{S}_{g,\Lambda}$) of a discrete (co-compact) subgroup of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

In short, the motivation for the current paper is the observation that the potentially computationally expensive task of computing the exact *canonical dual* or *tight Gabor atom* can be replaced by the very simple procedure of double preconditioning, which allows computing approximate dual or tight Gabor atoms "on the fly". Therefore, the concrete goal is to formulate the analytic background to this approach.

1.2. Paper Outline

The paper is structured as follows. In Section 2 we settle the preliminaries and notation. Section 3 introduces the basic ideas of the double preconditioning approach, followed by its formulation in the continuous setting, where we provide analytic formulas. Continuity questions are clarified in Section 5 and Section 4 builds the bridge to the spreading domain of operators. Then we consider double preconditioning in the context of Banach Gelfand Triples in Section 6. The last two sections provide further numerical insights, regarding the influence of the redundancy on the quality of preconditioning approaches and applications: finding an (approximate) tight window, improving the convergence of an iterative approach, and using it as a fast reconstruction option for varying lattices.

2. Preliminaries and Notation

In this paper, we will investigate both, the infinite-dimensional case, i.e. $f \in \boldsymbol{L}^2(\mathbb{R}^d)$, and the finite-dimensional case $f \in \mathbb{C}^n$, where $f = (f[0], f[1], \ldots, f[n-1])^T$ is always considered with indices to be modulo n, i.e. $f[k+ln] = f[k]$. In fact, we view finite vectors as signals defined on $\mathbb{Z}_n$, the cyclic group of order n, or as members of $\boldsymbol{\ell}^2(\mathbb{Z}_n)$. We will use the unitary version of the DFT (discrete Fourier transform), i.e. $f \mapsto \mathtt{fft}(f)/\sqrt{n}$.

To apply an operator O on the Fourier side we write $O^{\mathcal{F}} = \mathcal{F} \circ O \circ \mathcal{F}^*$, where $\mathcal{F}$ denotes the (unitary) Fourier transform, i.e. $O^{\mathcal{F}}$ describes O in the Fourier basis. As usual, we write $\mathcal{F}f = \widehat{f}$ and so $\mathcal{F}Of = O^{\mathcal{F}}\widehat{f}$.

For a given $m \times n$ matrix M (describing a linear mapping in the standard basis) one can write the same action in the Fourier basis via $M^{\mathcal{F}} = \{F_n M F_n^*$, where $(F_n)_{k,l} = (1/\sqrt{n}) \cdot e^{-(2\pi i k l/n)}$ is the DFT/FFT matrix, with $0 \leq k, l \leq n-1$.

Note that, for the matrix M, one could also use the two-dimensional Fourier transform $\mathcal{F}_2$, where $\mathcal{F}_2(M) = F_m M F_n$. And so the connection, described in terms of matrix entries, reads $\mathcal{F}_2(M)[k,l] = M^{\mathcal{F}}[k,-l]$.

2.1. Gabor Analysis

In this article, we will work with the triple (g,a,b), where $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ is the given *Gabor atom* (or window) and the constants $a,b>0$ describe the time-frequency lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, having lattice elements $\lambda = (ak, b\ell)$, with $k,\ell \in \mathbb{Z}^d$. By $T_\tau g(t) = g(t-\tau)$ we denote the *translation* or time-shift of g by τ and by $M_\xi g = g(t)e^{2\pi i \xi\cdot t}$ the *modulation* or frequency-shift of g by ξ. The family $(g_\lambda = M_{\ell b}T_{ka}g = \pi(\lambda)g)_{\lambda=(ka,\ell b)\in\Lambda}$ of all *time-frequency-shifted versions* of g, is called a *regular Gabor system*. To shorten the notation we will write (g,Λ) for such a collection of functions in $\boldsymbol{L}^2(\mathbb{R}^d)$. This notation is also convenient for more general lattices in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. It allows a stable reconstruction formula (or atomic decomposition with control of coefficients) if and only if the Gabor frame operator $\mathbf{S}_{g,\Lambda}$, given by

$$\mathbf{S}_{g,\Lambda}(f) = \sum_{\lambda\in\Lambda} \langle f, g_\lambda\rangle g_\lambda \tag{1}$$

is invertible. In fact, this is satisfied if and only if the Gabor system constitutes a frame [8], i.e. there exist constants $0 < A \le B < \infty$, such that

$$A\|f\|_2^2 = A\langle f,f\rangle \le \sum_{\lambda\in\Lambda} |\langle f, g_\lambda\rangle|^2 \le B\|f\|_2^2 \quad \forall\ f \in \boldsymbol{L}^2(\mathbb{R}^d) \tag{2}$$

holds. In this case, the *(canonical) dual frame* is the Gabor frame generated from the (canonical) *dual atom* $\widetilde{g}$ which is given by $\widetilde{g} = \mathbf{S}_{g,\Lambda}^{-1} g$, respectively the unique solution to the linear equation $\mathbf{S}_{g,\Lambda}\widetilde{g} = g$. This is a special instance where one can decompose any $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ as

$$f = \mathbf{S}_{g,\widetilde{g},\Lambda} f := \sum_{\lambda\in\Lambda} \langle f, g_\lambda\rangle \widetilde{g}_\lambda = \sum_{\lambda\in\Lambda} \langle f, \widetilde{g}_\lambda\rangle g_\lambda =: \mathbf{S}_{\widetilde{g},g,\Lambda} f. \tag{3}$$

In fact, the frame operator of the Gabor frame generated from the dual Gabor atom, $(\tilde{g}_\lambda)_{\lambda\in\Lambda}$ is just the inverse of the original Gabor frame operator, i.e.

$$\mathbf{S}_{g,\Lambda}^{-1}(f) = \mathbf{S}_{\tilde{g},\Lambda}(f) = \sum_{\lambda\in\Lambda} \langle f, \tilde{g}_\lambda\rangle \tilde{g}_\lambda. \tag{4}$$

Definition 2.1. *A function $\gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$ is called an approximate dual Gabor atom for a given pair (g, Λ) if the mixed Gabor operator*

$$\mathbf{S}_{g,\gamma,\Lambda}(f) = \sum_{\lambda\in\Lambda} \langle f, g_\lambda\rangle \gamma_\lambda \tag{5}$$

is close to the identity operator, i.e., if

$$\|\!|\!|\mathbf{S}_{g,\gamma,\Lambda} - \mathrm{Id}|\!|\!|_{\boldsymbol{L}^2\to\boldsymbol{L}^2} \leq \rho < 1. \tag{6}$$

For nice *Gabor atoms* g, the approximate duality is the better the smaller the parameter $\rho > 0$ is. Details and related concepts can be found in [43, 13, 12] or Def. 3.1. in [64].

One usually considers (for good reasons) an atom g used for a Gabor frame $(g_\lambda)_{\lambda\in\Lambda}$ to be "good" when it is in Feichtinger's Algebra, i.e.

$$g \in \boldsymbol{S}_0(\mathbb{R}^d) = \{f \in \boldsymbol{L}^2(\mathbb{R}^d) : V_g f \in L^1(\mathbb{R}^{2d})\}, \tag{7}$$

where

$$V_g f(t,\omega) = \langle f, \pi(t,\omega) g\rangle_{\boldsymbol{L}^2}, \quad (t,\omega) \in \mathbb{R}^2 \tag{8}$$

is the Short-Time Fourier Transform (STFT) of f with respect to the window g. Among others, this implies that any Gabor family arising from a general lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ defines a *Bessel family* for the Hilbert space $\big(\boldsymbol{L}^2(\mathbb{R}^d),\, \|\cdot\|_2\big)$. Moreover, according to well-established, non-trivial results, we know that the canonical dual Gabor atom $\widetilde{g}$ also belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$ (by [44]) and that $\widetilde{g} = \mathbf{S}_{g,\Lambda}^{-1}(g)$ depends continuously on the lattice parameters describing Λ (according to [34]). For these reasons, we will focus on atoms $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ and consider the operator norms also with respect to $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$, which dominates the operator norm on $\big(\boldsymbol{L}^2(\mathbb{R}^d),\, \|\cdot\|_2\big)$.

Except for the critical case ($a = b = 1$), a Gabor frame is non-orthogonal and redundant. Still, the Gabor frame operator has a lot of structure due to its invariance properties, and above all due to the fact that for any two functions $g, \gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$ one has

$$\mathbf{S}_{g,\gamma,\Lambda} \circ \pi(\lambda) = \pi(\lambda) \circ \mathbf{S}_{g,\gamma,\Lambda}, \quad \lambda\in\Lambda, \tag{9}$$

which is (together with the positive definiteness inherent to all frames) the basis for a variety of fast numerical algorithms to compute the (potentially non-canonical) *dual Gabor atom* numerically. The benefits of the *regular*

case, i.e. of the lattice structure of Λ becomes visible by two particular representations.

The *Walnut representation* (see [43], Chap.7) describes $\mathbf{S}_{g,\gamma,\Lambda}$ as strongly convergent series of translations combined with multiplications:

$$\mathbf{S}_{g,\gamma,\Lambda} = \sum_{l \in \mathbb{Z}^d} G_l \cdot T_{lb^{-1}}, \tag{10}$$

with G_l describing the multiplication operator by the a-periodic function

$$G_l(x) = b^{-d} \sum_{k \in \mathbb{Z}^d} \overline{g(x - lb^{-1} - ak)}\, \gamma(x - ak), \tag{11}$$

the so-called *correlation function* for the pair (g, γ). This representation goes back to [62] and has been explored numerically for a number of algorithms, as described in the early papers by S. Qiu ([55, 56, 54] to name a few). See also [14].

For our description also the *spreading representation* is relevant. It describes operators as (limits of) infinite series of time-frequency shifts [41]. In this sense, Equation (9) can be explained equivalently by the fact that the spreading representation of the Gabor frame operator is supported by another lattice, called the *adjoint lattice* $\Lambda^\circ = b^{-1}\mathbb{Z}^d \times a^{-1}\mathbb{Z}^d$, which can be characterized as the set of all time-frequency shifts commuting with those of Λ. This allows deriving the so-called *Janssen representation* of the Gabor frame operator, i.e. a description of $\mathbf{S}_{g,\gamma,\Lambda}$ as sum of TF-shifts from Λ°. In fact, the coefficients can be obtained explicitly (up to a factor depending on the redundancy of Λ) as samples of $V_g\gamma$ - the STFT of γ with respect to the window g - over Λ° (see [35], Thm. 3.5.11.iii):

$$\mathbf{S}_{g,\gamma,\Lambda} = \mathrm{red}(\Lambda) \sum_{\lambda^\circ \in \Lambda^\circ} V_g\gamma(\lambda^\circ)\pi(\lambda^\circ). \tag{12}$$

Although the Walnut representation was the original motivation for the preconditioning techniques, we will have a closer look at the Janssen representation in Section 4 to draw new functional analytic connections, yet, both perspectives play a role for the arguments provided. Therefore, we give an explicit description of the situation in the finite discrete case next.

2.1.1. The Matrix Viewpoint of the Gabor Frame Operator

An extensive introduction to discrete Gabor Analysis can be found in [60, 59]. In the finite discrete setting, we can only use lattice parameters

$a, b \in \mathbb{Z}$ such that $\frac{n}{a}$ and $\frac{n}{b}$ are integers. In matrix terminology, the (mixed) Gabor frame operator is then given by a $n \times n$ matrix $S_{g,\gamma,\Lambda}[m, \ell]$ where we use the indexing $m, l = 0, ..., n-1$.

The two mentioned representations of $S_{g,\gamma,\Lambda}$ above can be transferred to this finite setting as follows. Using Walnut's representation, one finds the entries of the Gabor frame operator matrix to be given by

$$\mathbf{S}_{g,\gamma,\Lambda}[m, \ell] = \begin{cases} \frac{n}{b} \sum_{k=0}^{\frac{n}{a}-1} \overline{g[m-ak]}\, \gamma[l-ak] & \text{for} \quad m - l \equiv 0 \mod \frac{n}{b} \\ 0 & \text{else.} \end{cases} \tag{13}$$

In other words, $\mathbf{S}_{g,\gamma,\Lambda}$ is a matrix that contains at most b non-zero side-diagonals (understood in a cyclic way), which are regularly spaced at distance n/b. Each of these side diagonals is a-periodic.

The Janssen representation can be used to fully describe $S_{g,\gamma,\Lambda}$ by a unique $a \times b$ matrix J with entries given by

$$J[m, l] = \frac{n}{a \cdot b} \cdot V_g \gamma \left[l\frac{n}{b}, m\frac{n}{a} \right]. \tag{14}$$

The rows of this (compact version) of the Janssen representation corresponds then to the (finite) Fourier transforms of the a-periodic functions found in the b side-diagonals.

2.1.2. Algebraic Prelude

Following the spirit of [63] or [58] the natural domain for Gabor Analysis is the realm of locally compact Abelian groups G, because it is based on time and frequency shifts and the use of the Short-Time Fourier Transform. An important subclass of such groups is, of course, the family of *finite* Abelian groups. As the dual group $\widehat{G}$ of such a group is also a finite Abelian group of the same cardinality, the associated *finite phase space* is the finite Abelian group $G \times \widehat{G}$. In contrast to the general case, and even compared to the usual setting, namely $G = \mathbb{R}^d$, the corresponding spaces (of functions or operators) are all *finite dimensional* and thus the issue of convergence of series or the boundedness of linear mappings can be ignored. Nevertheless, one can speak of well or poorly-conditioned problems.

Although every finite Abelian group is isomorphic to a product of cyclic groups of prime power order it is more elegant to provide a *coordinate-free* description of the situation from the perspective of abstract harmonic analysis as in [36]. We will follow the notations of that paper (also compatible with the standard notation of [43]) here.

The approach has the advantage that it avoids the use of multiple indices, even if one deals in practice with multi-dimensional objects, such as pixel images, which can be viewed as functions on $G = \mathbb{Z}_M \times \mathbb{Z}_N$ with a four-dimensional phase space and corresponding complicated lattices there. The same principle works for "matrices". The vector spaces can be treated by the methods of linear algebra and allow - at least in principle - an exact realization using the tools of matrix analysis. Instead of *frames* in infinite dimensional Hilbert spaces (such as $\left(\boldsymbol{L}^2(G),\, \|\cdot\|_2\right)$) one may consider finite sets which form a *generating family* for the finite-dimensional vector space under consideration. The approach to this viewpoint can be found in [38].

2.1.3. Banach Gelfand Triples

Let us now collect some functional analytic facts for this paper. It is quite natural to choose the Gabor atom g in $\boldsymbol{S}_0(\mathbb{R}^d)$. According to the important results in [32] (for the rational case) and [44] for the general case it can be assured, that in this case the dual window also belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$, and the same is true for the canonical tight window $h = \mathbf{S}_{g,\Lambda}^{-1/2} g$. These results are based on the principle of spectral invariance, which guarantees that for a Gabor frame operator of the form $\mathbf{S}_{g,\Lambda}$ the invertibility on $\left(\boldsymbol{L}^2(\mathbb{R}^d),\, \|\cdot\|_2\right)$ combined with the boundedness on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ automatically implies the invertibility as an operator on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, and hence (by duality) also on $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$. Such a result is by no means obvious, but unfortunately, this approach does not provide quantitative estimates for the inverse operators.

Over the years it has turned out to be convenient to describe the situation with the help of the *Banach Gelfand Triple* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ (going back to the chapters [41] and [35] in the book [40]). Comprehensive sources for this topic are [15] and [6]. Let us recall the definition:

Definition 2.2. *Let $\boldsymbol{B}$ be a Banach, and $\boldsymbol{H}$ a Hilbert space. The triple $(\boldsymbol{B},\boldsymbol{H},\boldsymbol{B}')$ forms a Banach Gelfand Triple if there are continuous embeddings*

$$\boldsymbol{B} \hookrightarrow \boldsymbol{H} \hookrightarrow \boldsymbol{B}' \tag{15}$$

which are dense in the first case and w^*-dense for the second.

The *prototypical* Banach Gelfand triple is $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$, with $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}}) = \left(\boldsymbol{\ell}^1,\, \|\cdot\|_1\right)$ as the basis, while the one interesting for us is $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$. One of the advantages of the specific setting of $\boldsymbol{S}_0$ is the fact that it can be defined over any LCA (locally compact Abelian) group [48], so it makes

sense to use not only $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$, but also $\boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ (viewing *phase space as a LCA group).*

In general, Banach Gelfand Triples constitute a *category* (see [1], [50]), where these triples form the objects. Even more important is the choice of the *morphism* in a category, i.e. the structure-preserving mappings between the objects in a category, called BGT-*morphism.*

Definition 2.3. *A linear mapping T between a given pair of Banach Gelfand Triples $(\boldsymbol{B}^1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}^2, \mathcal{H}_2, \boldsymbol{B}_2')$ is called a BGT-morphism if*

1. *T maps $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ to $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$,*
2. *The mapping T extends in a unique fashion to a bounded linear mapping from $(\mathcal{H}_1, \|\cdot\|_{\mathcal{H}_1})$ to $(\mathcal{H}_2, \|\cdot\|_{\mathcal{H}_2})$;*
3. *T extends furthermore to a norm* and *w^*-continuous mapping between the dual spaces, i.e. from $\big(\boldsymbol{B}_1', \|\cdot\|_{\boldsymbol{B}_1'}\big)$ to $\big(\boldsymbol{B}_2', \|\cdot\|_{\boldsymbol{B}_2'}\big)$.*

The norm of T as a BGT-mapping is the maximum over the three (operator) norms, i.e. $|||T|||_{(\boldsymbol{B},\mathcal{H},\boldsymbol{B}')} = \max\{|||T|||_{\boldsymbol{B}\to\boldsymbol{B}}\,, |||T|||_{\mathcal{H}\to\mathcal{H}}\,, |||T|||_{\boldsymbol{B}'\to\boldsymbol{B}'}\}$.

Correspondingly, the terms *BGT-isomorphism* (a morphism with the property that there is an inverse mapping, which is also a BGT-morphism) and the term *BGT-automorphism* are arising in a natural way. An isomorphism is called *unitary* if it is unitary at the Hilbert space level.

2.1.4. The Feichtinger BGTs

In most cases of interest the Hilbert space in the middle of a BGT is obtained by complex interpolation of Banach spaces between $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and $(\boldsymbol{B}', \|\cdot\|_{\boldsymbol{B}'})$ and thus the w^*-dense embedding of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ into $(\boldsymbol{B}', \|\cdot\|_{\boldsymbol{B}'})$ is the crucial assumption. Thus, the norm of a self-adjoint or unitary operator is just the norm on the inner Banach space (see [35] or [15]), i.e. $|||T|||_{(\boldsymbol{B},\mathcal{H},\boldsymbol{B}')} = |||T|||_{\boldsymbol{B}}$ in such a case. In particular, this is the situation we encounter for $T = \mathbf{S}_{g,\Lambda}$, with $\boldsymbol{B} = \boldsymbol{S}_0(\mathbb{R}^d)$. Let us formulate two well-known facts in the context of such Feichtinger BGTs:

1. The Fourier transform establishes an automorphism of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, which is uniquely determined by the fact that it maps pure frequencies χ_s to the corresponding Dirac measures δ_s;
2. The classical Fourier transform for periodic functions can be viewed as a Banach Gelfand Triple isomorphism between the BGT $(\boldsymbol{A}, \mathcal{H}, \boldsymbol{A}')$

and $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)(\mathbb{Z})$, i.e. the Gelfand triple consisting of Wiener's algebra $\left(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}}\right)$, the Lebesgue space $\left(\boldsymbol{L}^2(\mathbb{T}), \|\cdot\|_2\right)$ and the dual $(\boldsymbol{A}', \|\cdot\|_{\boldsymbol{A}'})$, also known as space of *pseudo-measures* on the torus, and the corresponding sequence spaces over $\mathbb{Z}$. Since one has $\boldsymbol{S}_0(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})$ due to the compactness of $\mathbb{T}$ and $\boldsymbol{S}_0(\mathbb{Z}) = \boldsymbol{\ell}^1(\mathbb{Z})$ due to the discreteness this is a first example of the general statement that $\mathcal{F}_G$ defines a unitary BGT-isomorphism between $\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0(G)}\right)$ and $\left(\boldsymbol{S}_0(\widehat{G}), \|\cdot\|_{\boldsymbol{S}_0(\widehat{G})}\right)$.

By the main result of [44] we know that for Gabor atoms $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ the Gabor frame operator defines not just an automorphism at the level of the Hilbert space (by definition) but extends to an automorphism of the Banach Gelfand Triple. For our purpose, we shall formulate this in the context of the Banach Gelfand Triple.

Theorem 2.4. *Given $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, such that (g, Λ) generates a Gabor frame for $\boldsymbol{L}^2(\mathbb{R}^d)$. Then $\mathbf{S}_{g,\Lambda}$ defines a BGT automorphism for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ with the inverse being the Gabor frame operator arising from $\widetilde{g}$.*

We note, however, that the abstract results established in this way, while far-reaching and exhausting for the case of the Gaussian or for totally positive functions (see [45]), do not provide bounds on the BGT-norms of the resulting isomorphism, respectively do not provide bounds for the inverse frame operator on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (or equivalently, on the $\boldsymbol{S}_0$-norm of $\widetilde{g}$). It is one of the functional analytic goals of this paper to provide some estimates on this norm, making use of the idea of preconditioning, see Section 6.

For this, we shall collect a few more known results. The first one is the formulation of the so-called *Janssen criterion*, i.e. the verification of sufficient conditions for a Gabor family generated from a pair (g, Λ). It is based on the fact, that the Janssen representation of the Gabor frame operator (see (12)) is an absolutely convergent sum of operators. For the description of this characterization, we refer to [41] (Thm.3.5.11) also known as the *Fundamental Identity for Gabor Analysis* (see [37]).

Combining general arguments for the inverse in Banach algebras with the Janssen criterion allows the following reformulation of [61, Theorem 4.11.]:

Theorem 2.5. *Given $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, with $\|g\|_2 = 1$ and $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. If*

$$\sum_{\lambda^\circ \in \Lambda^\circ} |V_g g(\lambda^\circ)| < 1 + \gamma, \tag{16}$$

for some $\gamma < 1$, then the frame operator $\mathbf{S}_{g,\Lambda}$ is invertible, and the operator norm on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (hence the BGT norm) of $\mathbf{S}_{g,\Lambda}^{-1}$ is controlled by

$$|||\mathbf{S}_{\widetilde{g},\Lambda}|||_{\boldsymbol{S}_0\to\boldsymbol{S}_0} = |||\mathbf{S}_{g,\Lambda}^{-1}|||_{\boldsymbol{S}_0\to\boldsymbol{S}_0} < \frac{1}{1-\gamma}. \tag{17}$$

This criterion was first formulated in [61, Theorem 4.11.]. It can be used as well to control the operator norm of $\mathbf{S}_{g,\Lambda}^{-1}$ on any of the spaces $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ (even uniformly with respect to $p \in [1,\infty]$), and even for other Banach space of functions or distributions which allow isometric action of the TF-shifts $\pi(\lambda)$.

To finish the preliminary part of the paper, let us point to the key argument of the claim above, i.e. an estimate of the difference between $\mathrm{Id}_{\boldsymbol{S}_0} = \pi(0,0) = V_g g(0,0)\pi(0,0)$ and $\mathrm{red}(\Lambda)\cdot\mathbf{S}_{g,\Lambda}$ (see Section 6).

3. The Double Preconditioning Method

The double preconditioning method was introduced in [4] as a way to compute approximate dual Gabor frames in a computationally very efficient way, and use that for an iterative approach to calculate the canonical dual. The idea is based on the fact that for relatively short windows the Gabor frame operator is reduced to a multiplication operator (called the *painless* case [17, 43, 3]). Therefore one may expect that in a more general situation with only good decay of the Gabor atom such a multiplication operator will be the dominant contribution to the Gabor frame operator. In fact, one has diagonal dominance and consequently invertibility of the Gabor frame matrix in the corresponding discrete case. On the other hand, it is also clear, that any (even a re-scaled) version of the Gabor frame operator cannot be close to the identity operator if the diagonal part shows strong oscillations. Hence getting rid of this problem by applying the inverse of the diagonal part of the Gabor frame operator is supposed to improve the situation significantly. This is the motivation for the first preconditioner, called *diagonal preconditioner*, denoted by $\mathfrak{D}_{g,\Lambda} := {\mathbf{D}_{g,\Lambda}}^{-1}$. Originally, the preconditioners were introduced in a finite-dimensional setting, such that $\mathbf{D}_{g,\Lambda}$ is the diagonal matrix consisting of the main diagonal of the Gabor frame matrix $\mathbf{S}_{g,\Lambda}$. We choose the notation in a way so that we can reuse it to introduce the continuous operator versions of the preconditioners conveniently later on.

By analogy, a good concentration of the Gabor atom in the frequency domain, i.e. smoothness of the Gabor atom has the same effect, but in the

Fourier transform description of the Gabor frame operator. Taken back to the time side, this is a situation where the side-diagonals in the Walnut representation of the Gabor frame operator are not negligible, but, by assumption, rather smooth periodic functions with small oscillations. Hence, they are well approximated by a convolution operator which can easily be inverted as well. This motivates us to perform the diagonal preconditioner on the Fourier side, which defines the *circulant* respectively *convolutional preconditioner*. In analogy to the diagonal case, we will use the symbol $\mathfrak{C}_{g,\Lambda}$, i.e. for $(\mathbf{C}_{g,\Lambda})^{-1}$, the inverse of the circulant operator which approximates $\mathbf{S}_{g,\Lambda}$.

Remark 3.1. *We denote the best approximation of* $\mathbf{S}_{g,\Lambda}$ *by a diagonal or a circulant matrix in the Frobenius norm by* $\mathbf{D}\left(\mathbf{S}_{g,\Lambda}\right)$ *and* $\mathbf{C}\left(\mathbf{S}_{g,\Lambda}\right)$*, respectively. One can obtain* $\mathbf{C}\left(\mathbf{S}_{g,\Lambda}\right)$ *as (best) approximation by a diagonal matrix on the Fourier side as introduced above. We would like to emphasize that the obtained convolution matrix can be constructed also directly without the Fourier representation and arises by taking the means of each side-diagonal of* $\mathbf{S}_{g,\Lambda}$*, see Appendix* 1*.*

Remark 3.2. *The idea to apply circulant preconditioners in order to speed up the iterative solvers for linear equations as such is not new and has been described in the literature many times. Let us just give two typical references, to [10] and [51].*

Both preconditioners are good candidates to approximate the inverse of the Gabor frame operator (for "nice" settings). In combination they are used to build the proposed double preconditioner $\mathfrak{P}_{g,\Lambda} = \mathfrak{P}(\mathbf{S}_{g,\Lambda})$ as

$$\mathfrak{P}_{g,\Lambda} = \mathbf{C}\left(\mathbf{D}\left(\mathbf{S}_{g,\Lambda}\right)^{-1} \cdot \mathbf{S}_{g,\Lambda}\right)^{-1} \mathbf{D}\left(\mathbf{S}_{g,\Lambda}\right)^{-1}, \tag{18}$$

which can be re-written to be a concatenation of operators

$$\mathfrak{P}_{g,\Lambda} = \mathfrak{C}_{\boldsymbol{A}} \circ \mathfrak{D}_{g,\Lambda}. \tag{19}$$

By $\mathfrak{C}_{\boldsymbol{A}} = \mathbf{C}\left(\mathfrak{D}_{g,\Lambda} \circ \mathbf{S}_{g,\Lambda}\right)^{-1}$ we denote a circulant preconditioner, which takes a diagonal preconditioner already into account. An approximate dual of g w.r.t. the lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$ can then be computed as

$$\widetilde{g}^{(ap)} = \mathfrak{P}_{g,\Lambda} g. \tag{20}$$

Note that this generates an approximate dual frame in the sense of [14]. Of course, the closeness of $\widetilde{g}^{(ap)}$ to $\widetilde{g}$ (in $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$) is intimately related to the closeness of $\mathfrak{P}_{g_0,\Lambda}$ to $\mathbf{S}_{g,\Lambda}^{-1}$ (as operator on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and hence on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$). For more see Section 5.

3.1. Preconditioning in the Continuous Setting

As mentioned, double preconditioning was introduced in [4] only for the finite-dimensional matrix setting. There, mostly numerical experiments were done showing that the approach works impressively well in many cases. In this section, we extend the method to the continuous setting using the Walnut representation of the Gabor frame operator, allowing explicit analytical formulas for the preconditioning operators and finding them to be convenient multiplication operators. The obtained formulas can also be used in the discrete setting to compute the approximate canonical dual Gabor atoms directly without explicitly constructing the full preconditioning matrices for fixed parameters.

For the diagonal preconditioner, note that the main diagonal of $\mathbf{S}_{g,\Lambda}$ is given by the correlation function G_l in the Walnut representation of $\mathbf{S}_{g,\Lambda}$ (see Equation (10)) for $l = 0$, yielding an a-periodized version of $|g|^2$,

$$G_0(t) = b^{-d} \sum_{k\in\mathbb{Z}^d} |g(t - ak)|^2. \tag{21}$$

For the circulant preconditioning approach we use the identity $\mathbf{S}_{g,\Lambda}^{\mathcal{F}} = \mathbf{S}_{\widehat{g},\Lambda'}$ where $\Lambda' = b\mathbb{Z} \times a\mathbb{Z}$ and find the main diagonal of $\mathbf{S}_{g,\Lambda}^{\mathcal{F}}$ analogously, denoting the correlation term on the Fourier side as

$$G_0^{\mathcal{F}}(s) = a^{-d} \sum_{\ell\in\mathbb{Z}^d} |\widehat{g}(s - b\ell)|^2. \tag{22}$$

In this sense, the two single preconditioning operators can be written directly as multiplication operators in the continuous setting,

$$\mathfrak{D}_{g,\Lambda} f(t) = \mathbf{D}_{g,a}^{-1} f(t) = G_0^{-1}(t) \cdot f(t) \tag{23}$$

$$\mathfrak{C}_{g,\Lambda} f(s) = \mathbf{C}_{g,b}^{-1} f(s) = \left[\mathcal{F}^* \left((G_0^{\mathcal{F}})^{-1}\right)\right](s) \cdot f(s). \tag{24}$$

In order to establish the double preconditioning operator we have to discuss it for the mixed version of the Gabor frame operators $\mathbf{S}_{g,\gamma}$. The resulting

diagonal version is then given by

$$\mathbf{D}_{g,\gamma,\Lambda} f(t) = \mathbf{D}\left(\mathbf{S}_{g,\gamma,\Lambda}\right) f(t) = \left[b^{-d} \sum_{k \in \mathbb{Z}^d} \overline{g(t-ak)} \gamma(t-ak) \right] \cdot f(t) \tag{25}$$

and the circulant mixed version by

$$\mathbf{C}^{\mathcal{F}}_{g,\gamma,\Lambda} = \mathcal{F} \circ \mathbf{D}_{\hat{g},\hat{\gamma},\Lambda} \circ \mathcal{F}^* = \mathbf{D}_{\hat{g},\hat{\gamma},\Lambda'}. \tag{26}$$

In this way, the mixed circulant preconditioner results in a convolution operator defined by the transfer function $\mathbf{C}^{\mathcal{F}}_{g,\gamma,\Lambda}$, i.e. we obtain a multiplication operator on the Fourier side,

$$\mathbf{C}^{\mathcal{F}}_{g,\gamma,\Lambda} \widehat{f}(s) = \left[a^{-d} \sum_{\ell \in \mathbb{Z}^d} \overline{\widehat{g}(s-b\ell)} \widehat{\gamma}(s-a\ell) \right] \cdot \widehat{f}(s). \tag{27}$$

In order to construct the double preconditioner we have to consider the main diagonal of $\mathbf{D}^{-1}_{g,\gamma,\Lambda} \mathbf{S}_{g,\gamma,\Lambda}$ on the Fourier side. In fact, this can be written as the mixed frame operator $\mathbf{S}_{g,\gamma^{\mathfrak{D}},\Lambda}$ with $\gamma^{\mathfrak{D}} = \mathfrak{D}_{g,\Lambda}\gamma$ being the diagonal preconditioned atom. This works because by (25) $\mathbf{D}^{-1}_{g,\gamma,\Lambda}$ is a multiplication operator and, therefore, commutes with all modulations. It is also a-periodic and therefore commutes with translations T_{ka} for all k. At a more abstract level, one could argue with the Janssen representation (12) that

$$\mathfrak{D}_{g,\gamma,\Lambda} = \mathrm{red}(\Lambda) \sum_{(0,\nu) \in \Lambda^\circ} \widehat{(\gamma \cdot \overline{g})}(\nu) M_\nu \tag{28}$$

and that it commutes therefore with all time-frequency shifts in Λ.

Therefore, we set $\mathfrak{C}_{g,\gamma^{\mathfrak{D}},\Lambda} = \mathbf{C}^{-1}_{g,\gamma^{\mathfrak{D}},\Lambda}$, describing the circulant preconditioner, which takes the previous diagonal preconditioning already into account. Then, using (25) and (27) we find the double preconditioning operator to be the composition

$$\mathfrak{P}_{g,\gamma,\Lambda} = \mathfrak{C}_{g,\gamma^{\mathfrak{D}},\Lambda} \circ \mathfrak{D}_{g,\gamma,\Lambda}. \tag{29}$$

For $g = \gamma$, the Walnut representation on the Fourier side has a correlation function for $l = 0$, given by

$$G^{\mathcal{F},\mathfrak{D}}_0(s) = a^{-d} \sum_{\ell \in \mathbb{Z}^d} \hat{g}(s-b\ell) \widehat{g^{\mathfrak{D}}}(s-b\ell). \tag{30}$$

Using this we can write the double preconditioning operator $\mathfrak{P}_{g,\Lambda}$ as the multiplication operator with a double factor

$$\mathfrak{P}_{g,\Lambda} f = \mathcal{F}^* \frac{1}{G_0^{\mathcal{F},\mathfrak{D}}} \cdot \mathcal{F} \frac{1}{G_0} \cdot f. \tag{31}$$

Doing the computations in a finite setting according to these formulas has two advantages. On the one hand, one gets rid of the necessary divisibility condition - i.e. that one has to enforce a and b being divisors of n - and can construct good approximate duals by choosing the lattice parameters more freely. See Section 8.3 for a direct application. On the other hand, the underlying computations can be done in a very fast manner, even outperforming the block processing scheme originally proposed in [4]. By using the block processing scheme [53] building the Gabor frame operators takes bn/a multiplications, in contrast to Equation (25), which takes only order n/a multiplications.

3.2. Almost Commutation of Double Preconditioning

Experimental evidence from many simulations indicates that the order of applying diagonal and circulant preconditioning does not matter so much in most settings [4]. This is obvious for the case of commutative TF-lattices since then both preconditioners have a Janssen representation with non-zero terms using only elements of Λ°, but this optimal situation occurs only rarely. For generic situations a more detailed investigation of the level of closeness (for lattices that are close to commutative ones and for "nice" Gabor atoms) still has to be done.

What the observations from numerous numerical experiments indicate is the following: If the first preconditioner is working well, then the second would not spoil the positive effect of the first one. In some cases, it may even significantly contribute to and improve the situation. On the other hand, if the more efficient one is applied as the second one it appears that doing the less efficient first does not deteriorate the overall effect. All these claims are expected to be valid for Gabor atoms g which are well concentrated in the TF-sense.

Using the correlation functions we can verify a certain form of commutativity for explicit conditions on the window functions g, γ, and the lattice Λ. For the *alternative* double preconditioner, which arises by first doing

convolutional preconditioning and then diagonal preconditioning we have:

$$\mathfrak{R}_{g,\Lambda} = \mathbf{D}\left(\mathfrak{C}_{g,\Lambda} \circ \mathbf{S}_{g,\Lambda}\right)^{-1} \circ \mathfrak{C}_{g,\Lambda} \tag{32}$$

analogously to (31), and thus

$$\mathfrak{R}_{g,\Lambda} f = \frac{1}{G_0^{\mathfrak{C}}} \cdot \mathcal{F}^* \frac{1}{G_0^{\mathcal{F}}} \cdot \mathcal{F} f, \tag{33}$$

where

$$G_0^{\mathfrak{C}}(t) = b^{-d} \sum_{k \in \mathbb{Z}^d} g(t - ak)\, g^{\mathfrak{C}}(t - ak) \tag{34}$$

is the correlation function for $\mathbf{S}_{g,g^{\mathfrak{C}},\Lambda}$ at $l = 0$ and $g^{\mathfrak{C}} = \mathfrak{C}_{g,\Lambda} g$.

We will show the commutation result for the more general mixed version, i.e. using two windows. Using (25) and (27) we get

$$\mathfrak{R}_{g,\gamma,\Lambda} = \mathfrak{D}_{g,\tilde{\gamma}^{\mathfrak{C}},\Lambda} \circ \mathfrak{C}_{g,\gamma,\Lambda}, \tag{35}$$

analogously to (29).

Theorem 3.3. *If g, γ are even symmetric windows, then*

$$\mathfrak{R}^{\mathcal{F}}_{g,\gamma,\Lambda} = \mathfrak{P}_{\hat{g},\hat{\gamma},\Lambda'}. \tag{36}$$

Proof. Noting that $\widehat{(\gamma^{\mathfrak{C}})} = (\hat{\gamma})^{\mathfrak{D}}$ we obtain

$$\mathfrak{R}^{\mathcal{F}}_{g,\gamma,\Lambda} = \mathfrak{D}^{\mathcal{F}}_{g,\tilde{\gamma}^{\mathfrak{C}},\Lambda} \circ \mathfrak{C}^{\mathcal{F}}_{g,\gamma,\Lambda} = \mathfrak{D}^{\mathcal{F}}_{g,\tilde{\gamma}^{\mathfrak{C}},\Lambda} \circ \mathfrak{D}_{\hat{g},\hat{\gamma},\Lambda'}. \tag{37}$$

Further, by (26) we have that

$$\mathfrak{D}^{\mathcal{F}}_{g,\gamma,\Lambda} = \mathcal{I}\mathfrak{C}_{\check{g},\check{\gamma},\Lambda'}\mathcal{I}, \tag{38}$$

where $\mathcal{I}$ is the reflection operator, i.e. $(\mathcal{I}f)(t) = f(-t)$. Indeed, for any operator O which has a Janssen representation as in (12) we have that $\mathcal{I}O\mathcal{I} = O$, as $-\Lambda^\circ = \Lambda^\circ$. Hence

$$\mathfrak{R}^{\mathcal{F}}_{g,\gamma,\Lambda} = \mathfrak{C}_{\mathcal{I}\hat{g},\mathcal{I}\hat{\gamma}^{\mathfrak{D}},\Lambda'} \circ \mathfrak{D}_{\hat{g},\hat{\gamma},\Lambda'}. \tag{39}$$

Assuming even windows, clearly $\mathcal{I}\hat{g} = \hat{g}$ holds, but also $\mathcal{I}\hat{\gamma}^{\mathfrak{D}} = \hat{\gamma}^{\mathfrak{D}}$ since

$$\mathcal{I}\widehat{\gamma^{\mathfrak{D}}} = \mathcal{I}\mathfrak{D}_{g,\gamma,\Lambda}\gamma = (\mathcal{I}\mathfrak{D}_{g,\gamma,\Lambda}\mathcal{I})\,\mathcal{I}\gamma = \mathfrak{D}_{g,\gamma,\Lambda}\mathcal{I}\gamma. \tag{40}$$

Therefore,

$$\mathfrak{R}^{\mathcal{F}}_{g,\gamma,\Lambda} = \mathfrak{P}_{\hat{g},\hat{\gamma},\Lambda'}, \tag{41}$$

and, moreover, also

$$\mathfrak{R}_{g,\gamma,\Lambda} = \mathfrak{P}^{\mathcal{F}}_{\hat{g},\hat{\gamma},\Lambda'}. \tag{42}$$

□

Corollary 3.4. *In the particular case of $g = \gamma$ being the standard Gaussian and assuming lattice parameters such that $\Lambda = \Lambda'$, one has*

$$\mathfrak{R}_{g,\gamma,\Lambda} = \mathfrak{P}^{\mathcal{F}}_{g,\gamma,\Lambda}. \tag{43}$$

In other words, interchanging the order of circulant and diagonal preconditioning in the double preconditioning operator corresponds to switching to the Fourier side of the operator. Hence, in this case, one has

$$\mathfrak{R}_{g,\gamma,\Lambda} g = \widehat{\mathfrak{P}_{g,\gamma,\Lambda} g}. \tag{44}$$

Note that the converse of the corollary is not true. As an example take two different windows such that their Gabor systems result in Parseval frames (for the same lattice), then all the involved preconditioning operators are the identity, but the windows do not coincide.

4. General Subgroup Preconditioning in the Spreading Domain

Let us go back to the finite-dimensional, yet, abstract group-motivated setting described at the beginning, where the Gabor frame operator lives in $\mathcal{M}_{n,n}$, the vector space of all complex-valued $n \times n$-matrices, which can be identified with the vector space of all functions on the finite phase-space is $\mathbb{Z}_n \times \widehat{\mathbb{Z}_n}$.

The Janssen representation shows - even in the most general situation - that the Gabor frame operator $\mathbf{S}_{g,\Lambda}$ derived from a Gabor atom $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ is an absolutely convergent sum of TF-shifts from $\Lambda^\circ \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. The connection to the idea of (double) preconditioning can be described as follows: The lattice points of Λ° on the y-axis represent all pure modulations. Hence, the main diagonal of the Gabor frame operator arises by taking exactly those elements, see (28). Indeed, all the other elements involve some translation (by shifts of multiples of a) and thus do not contribute to the main diagonal. Therefore, the diagonal preconditioner is directly related to the y-axis on the

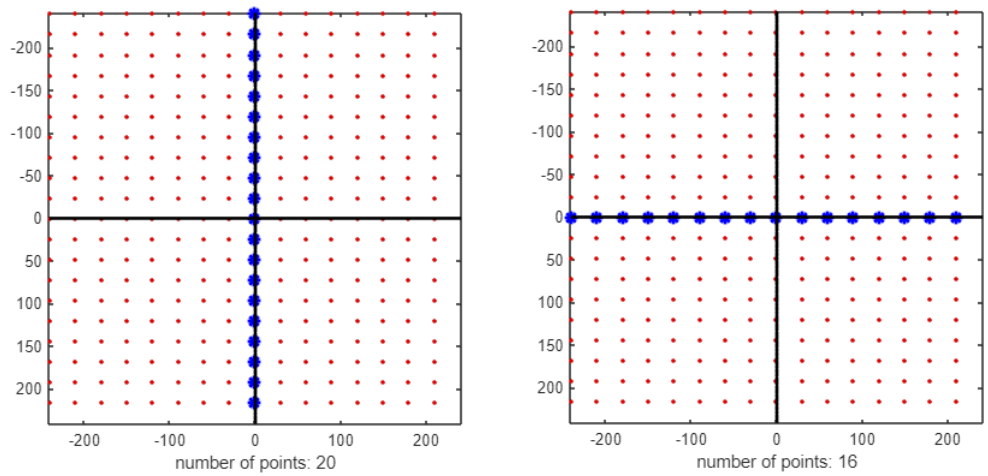

Figure 1: Left: Position of the Janssen coefficients on Λ° corresponding to the diagonal preconditioner. Right: Positions corresponding to the convolutional preconditioner.

adjoint lattice. As the adjoint lattice is given by $b^{-1}\mathbb{Z}^d \times a^{-1}\mathbb{Z}^d$, the Fourier transform of the main diagonal is concentrated on a lattice of the form $a^{-1}\mathbb{Z}^d$. As a consequence the main diagonal of $\mathbf{S}_{g,a,b}$ has to be an a-periodic function, in fact, we know it is an a-periodization of $|g|^2$. A similar statement can be given concerning the preconditioner, which arises from the lattice points of Λ° on the x-axis. Figure 1 illustrates this perspective by showing the positions of the corresponding Janssen coefficients within the lattice Λ°.

Let us provide a more abstract perspective of the situation (for the general setting). We consider $\mathcal{M}_{n,n}$ of all the complex-valued $n \times n$-matrices with the usual scalar product on $\mathbb{C}^{n^2}$. This is also known as the Frobenius product and corresponds to the scalar product defined in the Hilbert-Schmidt sense using the trace operator. Let us give a summary of known, respectively easily shown results:

Lemma 4.1. *Given a subgroup Λ of the finite phase space $\mathbb{Z}_n \times \widehat{\mathbb{Z}_n}$, with adjoint*[1] *group Λ° (see [35], 3.5.3) one has:*

1. *The subalgebra $\mathcal{M}_\Lambda \subset \mathcal{M}_{n,n}$, consisting of all $n \times n$-matrices which*

[1]It is just the commutator group in the TF-sense.

commute with the TF-shifts $\pi(\Lambda) := \{\pi(\lambda), \lambda \in \Lambda\}$*, is exactly the space of all matrices that have a spreading support contained in* Λ°*.*

2. *The (rescaled) TF-shift operators* $\pi(\lambda)/\sqrt{n}, \lambda \in \mathbb{Z}_n \times \widehat{\mathbb{Z}_n}$ *form an orthonormal basis for* $\mathcal{M}_{n,n}$*.*
   Consequently, the Hilbert-Schmidt (or Frobenius) norm is equivalent to the norm of its coefficients in $\boldsymbol{\ell}^2(\Lambda^\circ)$*. The corresponding expansion of a given operator* T *acting on* $\boldsymbol{\ell}^2(\mathbb{Z}_n)$*, i.e. the* $n \times n$*-matrix which describes (the coefficients of) the representation of* T *with respect to this ONB is called the* spreading representation *of* T*, denoted by* $\eta(T)$*.*
3. *Let* H *be a subgroup of* $\mathbb{Z}_n \times \widehat{\mathbb{Z}_n}$*. Then the linear span of* $\pi(H)$ *forms a Banach algebra if the coefficients are endowed with the norm of* $\boldsymbol{\ell}^1(H)$*. At the coefficient level, the composition of operators can be described in terms of twisted convolutions, i.e.*

$$(M \natural_\Lambda N)[k,l] = \sum_p \sum_q M[p,q] \cdot N[k-p, l-q] \cdot e^{\frac{2\pi i (k-p)a \cdot qb}{n}}.$$

4. *If* H *is a subgroup of* $\mathbb{Z}_n \times \widehat{\mathbb{Z}_n}$ *with the property that the TF-shifts arising from* H *commute*[2]*, then twisted convolution in* $\boldsymbol{\ell}^1(H)$ *is reduced to ordinary convolution, and thus the inversion of an operator in such a Banach algebra can be obtained by means of the pointwise division of the corresponding Fourier transform for* H*.*

Proof. 1.) The proof can be taken from the continuous setting, see e.g. [43, Lemma 7.4.1], or for the general version see [35].

2.) This can be found e.g. in [36].

3.) Given $O_i = \sum_{h_i \in H} \eta(O_i)(h_i)\pi(h_i)$ for $i = 1, 2$. Then let $h = (t_h, \nu_h)$ and

$$\begin{aligned} O_1 O_2 &= \sum_{h_1 \in H} \eta(O_1)(h_1)\pi(h_1) \sum_{h_2 \in H} \eta(O_2)(h_2)\pi(h_2) \\ &= \sum_{h_1 \in H} \sum_{h_2 \in H} \eta(O_1)(h_1)\eta(O_2)(h_2) e^{\frac{2\pi i t_{h_1} \nu_{h_2}}{n}} \pi(h_1 + h_2) \\ &= \sum_{h \in H} \left(\sum_{h' \in H} \eta(O_1)(h - h')\eta(O_2)(h') e^{\frac{2\pi i \left(t_h - t_{h'}\right)\nu_{h'}}{n}} \right) \pi(h). \end{aligned}$$

4.) Obvious by above. □

[2] We then call H a *TF-commutative* lattice in phase space.

Up to now, we have focused very much on Gabor frames generated by a triple (g, a, b), where the time and the frequency axis play a particular role. Their orthogonal lattices, i.e. the one of $a\mathbb{Z}^d$ is $1/a\mathbb{Z}^d$ in the frequency direction, and $1/b\mathbb{Z}^d$ in the time direction give us insight into a group theoretical interpretation of the two pre-conditioners, which will now be taken to a more general level. This will allow using the same idea in the context of general lattices $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. The obvious proof of the next lemma is left to the reader:

Lemma 4.2. *Let H be a subgroup of the TF-lattice Λ°, with adjoint group H°. Then the set of all matrices whose spreading symbol is supported on H is a subalgebra (with respect to twisted convolution), and will be denoted by $\mathcal{M}_H$. It corresponds to the subset of $\mathcal{M}_{n,n}$ of all $n \times n$-matrices with the property that they are invariant under conjugation with $\pi(h^\circ)$, with $h^\circ \in H^\circ$, i.e. satisfying*

$$\pi(h^\circ) \circ M = M \circ \pi(h^\circ). \tag{45}$$

Then the restriction of the spreading symbol of a given operator $T \in \mathcal{M}_\Lambda$ to H provides the best approximation (in the Hilbert-Schmidt sense) of T by elements from the sub-algebra $\mathcal{M}_H$. The best approximation of a given matrix M by an element from $\mathcal{M}_H$ can thus be described easily at the level of the spreading symbols: it is simply the pointwise multiplication with indicator function of H (resp. the Dirac comb over H).

For the particular case of diagonal and circulant preconditioners, the corresponding subgroups are $H = \{0\} \times \frac{1}{a}\mathbb{Z}_n$ and $H = \frac{1}{b}\mathbb{Z}_n \times \{0\}$.

Now, one may describe the general idea of (simple, double or other) preconditioning of the Gabor frame operator by choosing the most relevant, and perhaps most simple and most efficient commutative subgroup of Λ°, and use the inverse of the best approximation of the Gabor frame operator on this subgroup as a preconditioner.

Ideally, such an approach should fulfill the following properties:

1. It should be easy to build the best approximation B of S;
2. It should be easy to compute the inverse B^{-1} of the approximation;
3. The product $B^{-1}S$ should belong to the same Banach algebra of matrices (i.e. to those having a spreading support inside of Λ°).

The first two items are conditions one would need to have good preconditioning in the general case, while the third one would give a way to determine

the concatenation of those operators in the particular structure, which allows investigation as in Section 3.2.

Of course, there are many potential subgroups for this approach. Of particular interest are subgroups of Λ° with the property that the arising set of TF-shifts $\pi(\Lambda^\circ) = \{\pi(\lambda^\circ) \,|\, \lambda^\circ \in \Lambda^\circ\}$ forms a commutative group of operators. By Lemma 4.1 the composition of operators at the level of their spreading function is then equivalent to the ordinary convolution of functions over $G \times \widehat{G}$ and thus, can be inverted easily using Fourier methods. Therefore, condition (2) is fulfilled. In the non-commutative case, which occurs e.g. for small (non-integer) redundancies, the situation requires dealing with the inversion of the twisted convolution, which is a much harder problem, see [21]. Since twisted convolution introduces some form of non-commutativity of the involved operators it is natural to look out for *TF-commutative* subgroups which can be used to constitute cheap and (as it turns out) effective preconditioners. The subgroups that were chosen for the double preconditioning approach fulfill exactly this condition.

5. Continuous Dependence on the Parameters

It is a well-established result that the dual atom of a Gabor frame generated by an atom in $\boldsymbol{S}_0(\mathbb{R}^d)$ depends continuously on both the atom and the lattice Λ (see [33]). Therefore, it is natural to ask whether the same is true for the approximate dual Gabor atom generated by the double preconditioning method. This is not only a question of academic interest but also of practical relevance, e.g. for the discussion in Section 8.3. As a motivation for our expectation (a proof will be given below), Figure 2 shows the result of a numerical experiment, which indicates the continuous dependence of the shapes of the preconditioned windows from the lattice parameters. The length of the signal is $n = 480$ and the lattice parameters, both in time and frequency change from 15 to 18. Note that some of these parameters are *not divisors* of n, so the use of the representation of the double preconditioner as multiplication operator derived in Section 5 gets crucial.

We refer to [24, 41, 48] for general results on $\boldsymbol{S}_0(\mathbb{R}^d)$. We also use standard results concerning Wiener amalgam spaces $\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^q)$, see e.g. [26], [46] or [43]. Loosely speaking they allow controlling the *global* behaviour (summability) of the *local norm* expressed using $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$.

We will start our theoretical investigation with the following general result. Note that only the $\boldsymbol{S}_0$-norm allows estimating Bessel bounds for Gabor

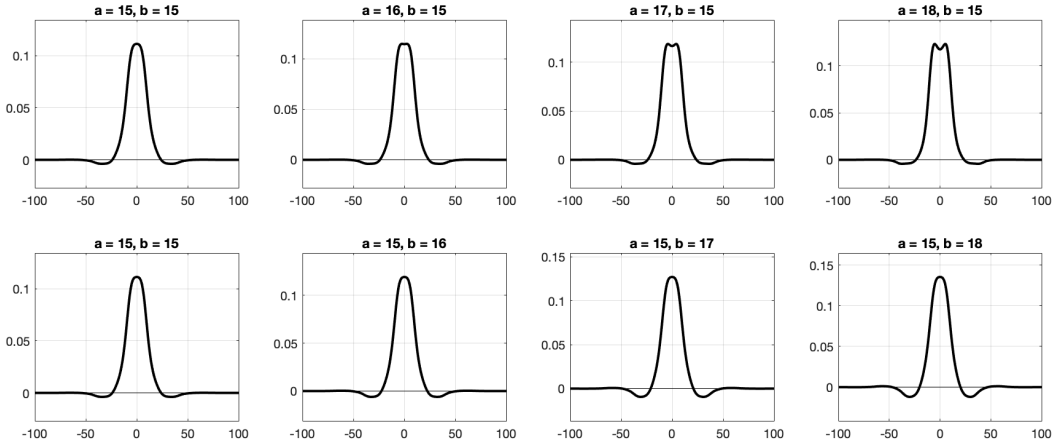

Figure 2: *Upper row*: Evolution of $\tilde{g}^{(ap)}$ for increasing a from 15 to 18. *Lower row:* Evolution of $\tilde{g}^{(ap)}$ for increasing b from 15 to 18.

families, uniformly over certain ranges of parameters. Hence, even if we would be only interested in operators between Hilbert spaces, the use of $\boldsymbol{S}_0$-estimates appears to be crucial (see [41], Section 3.3.3).

Proposition 5.1. *Let $\alpha = 1/a$ and $h_0 \in \boldsymbol{S}_0(\mathbb{R}^d)$ with $\widehat{h}_0(0) = 1$ and*

$$\sum_{k\in\mathbb{Z}^d} |\widehat{h}_0(\alpha_0 k)| = 1 + \gamma' < 2 \text{ for some } \alpha_0 > 0,\, \gamma' > 0.$$

Choose any $\gamma \in (\gamma', 1)$ some $\varepsilon' > 0$ and $\delta' > 0$, any $h \in \boldsymbol{S}_0(\mathbb{R}^d)$ with $\widehat{h}(0) = 1$, $\|h - h_0\|_{\boldsymbol{S}_0} \leq \varepsilon'$ and any $|\alpha - \alpha_0| < \delta'$. Then we note that the Fourier coefficients of the a-periodized version of h, i.e.

$$H(x) := \sum_{k\in\mathbb{Z}^d} h(x - ka) \quad \forall x \in \mathbb{R}^d, \tag{46}$$

satisfy the estimate

$$\sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\widehat{H}(k)| = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\widehat{h}(\alpha k)| < \gamma. \tag{47}$$

Consequently, we have

$$|H(x)| = \left| \sum_{k\in\mathbb{Z}^d} h(x - ka) \right| \geq 1 - \gamma > 0 \quad \forall x \in \mathbb{R}^d. \tag{48}$$

Hence, $1/H$ defines a non-vanishing function in $\boldsymbol{C}_b(\mathbb{R}^d)$, and the pointwise multiplication operators, given as $f \mapsto f/H$, are well defined. In fact, they form a uniformly bounded family of bounded operators on $\boldsymbol{S}_0(\mathbb{R}^d)$ with:

$$\|f/H\|_{\boldsymbol{S}_0} \leq (1-\gamma)^{-1}\|f\|_{\boldsymbol{S}_0}, \quad f \in \boldsymbol{S}_0(\mathbb{R}^d). \tag{49}$$

Proof. We start by observing that $h_0 \in \boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^1)(\mathbb{R}^d)$ implies that $\widehat{h}_0 \in \boldsymbol{S}_0(\mathbb{R}^d)$. Next we note that the dilation operators mapping $f \mapsto \mathrm{D}_\rho(f)$ with $\mathrm{D}_\rho(f)(x) = f(\rho x)$, are uniformly bounded on $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ for any compact range of parameters inside of $(0,\infty)$. By the restriction property for $\boldsymbol{S}_0(\mathbb{R}^d)$ the restriction mapping $R_{\mathbb{Z}^d}$ is bounded from $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ to $\boldsymbol{S}_0(\mathbb{Z})$, i.e. to $\big(\boldsymbol{\ell}^1(\mathbb{Z}^d), \|\cdot\|_{\boldsymbol{\ell}^1(\mathbb{Z}^d)}\big)$ is bounded, and thus $f \mapsto (f(\alpha k)_{k\in\mathbb{Z}^d})$ is a uniformly bounded family of operators from $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ to $\big(\boldsymbol{\ell}^1(\mathbb{Z}^d), \|\cdot\|_{\boldsymbol{\ell}^1(\mathbb{Z}^d)}\big)$. Together this implies that the mapping $S : (h,\alpha) \mapsto R_{\mathbb{Z}^d}(\mathrm{D}_\alpha \widehat{h})$ is a continuous mapping from $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big) \times [\alpha_0/2, 2\alpha_0]$ to $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$, which satisfies at (h_0,α_0) the condition

$$\|S(h_0,\alpha_0) - \delta_0\|_{\boldsymbol{\ell}^1(\mathbb{Z}^d)} < \gamma' < 1$$

and thus we will have, for $\gamma > \gamma'$:

$$\|S(h,\alpha) - \delta_0\|_{\boldsymbol{\ell}^1(\mathbb{Z}^d)} < \gamma < 1$$

for all pairs (h,α) which are close enough to the starting point (h,α_0), which is the same as the validity of (47).

In order to connect this observation with the statements of the proposition, we have to observe that each periodic function H appearing above belongs to Wiener's algebra $\boldsymbol{A}(\mathbb{T}^d)$ of absolutely convergent Fourier series over $\mathbb{T}^d$. In fact, this claim can be justified as follows. We have to show that the Fourier coefficients of H belong to $\boldsymbol{\ell}^1(\mathbb{Z}^d)$. Writing $\text{⨿}_a := \sum_{k\in\mathbb{Z}^d} \delta_{ak}$ we have:

$$\mathcal{F}(H) = \mathcal{F}(\text{⨿}_a * h) = C_a(\text{⨿}_\alpha \cdot \widehat{h}), \tag{50}$$

where C_a is a constant depending only on a (and the general setup). Equivalently, the Fourier coefficients of the periodic function H are just the samples of $\widehat{h}$ over the dual lattice, i.e. they are of the form $h \mapsto S(h,\alpha)$ for $\alpha = 1/a$, which form a family of bounded linear mappings from $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ to $\big(\boldsymbol{\ell}^1(\mathbb{Z}^d), \|\cdot\|_{\boldsymbol{\ell}^1(\mathbb{Z}^d)}\big)$. It is important to note that Wiener's algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$

is just the $\boldsymbol{\ell}^1$-norm of the Fourier coefficients of the absolutely convergent Fourier series and thus we have *independently from the lattice constant a:*

$$\|H\|_\infty \leq \|H\|_{\boldsymbol{A}}\,, \tag{51}$$

and thus $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is continuously embedded into $\big(\boldsymbol{C}_b(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$.

Now the estimate (47) implies that H is free of zeros and thus invertible as a bounded, continuous functions. But it is also invertible in the Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ (with convolution in $\boldsymbol{\ell}^1(\mathbb{Z}^d)$ as multiplication corresponding to point-wise multiplication of the functions H), and close to the identity Id, which corresponds to δ_0, the Dirac delta at 0 in the Fourier series expansion.

Note that we do not have to invoke Wiener's inversion theorem (see [57]), which would not allow giving a norm estimate of the inverse anyway, but we make use of the generic invertibility argument in general Banach algebras: If an element in a (here commutative) Banach algebra is close enough to the identity element in the given Banach algebra, i.e.

$$\|1 - H\|_{\boldsymbol{A}} = \|\delta_0 - (\widehat{h}(bk))_{k\in\mathbb{Z}^d}\|_{\boldsymbol{\ell}^1(\mathbb{Z}^d)} = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\widehat{h}(bk)| < \gamma,$$

then the element is invertible and

$$\|1/H\|_{\boldsymbol{A}} = \|\sum_{n=0}^{\infty}(1-H)^n\|_{\boldsymbol{A}} \leq (1-\gamma)^{-1}.$$

Since the modulation operators (multiplication with pure frequencies) are isometric on $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ (if we use the standard norm) it follows that (49) is valid:

$$\|f/H\|_{\boldsymbol{S}_0} \leq \|1/H\|_{\boldsymbol{A}}\, \|f\|_{\boldsymbol{S}_0} \leq (1-\gamma)^{-1}\|f\|_{\boldsymbol{S}_0},\ f \in \boldsymbol{S}_0(\mathbb{R}^d). \tag{52}$$

□

Given $h_0, \alpha_0, a_0, \varepsilon'$ and $\delta > 0$ as above we have:

Proposition 5.2. *For any sequence $a_n \to a_0$ and $h_n \to h_0$ in $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$:*

$$\lim_{n\to\infty} \|f/H_n - f/H_0\|_{\boldsymbol{S}_0} = 0, \quad f \in \boldsymbol{S}_0(\mathbb{R}^d). \tag{53}$$

Proof. The continuous dependence on $h \in \left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is already covered by Proposition 5.1.

Concerning the dependence on the lattice constant a we show strong convergence. For this consider a fixed $f \in \boldsymbol{S}_0(\mathbb{R}^d)$. Due to the uniform boundedness of the family of division operators we can restrict our attention to a dense subspace of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$. It is convenient to assume in this step (without loss of generality) that f is band-limited, i.e. that $\operatorname{supp}(\widehat{f}) \subset B_R(0)$ for some $R > 0$.

As we have seen, we can write the a-periodized version H of h as

$$\text{⨿}_a * h = \left(\sum_{k\in\mathbb{Z}^d} \delta_{ak}\right) * h = \sum_{k\in\mathbb{Z}^d} T_{ak} h = H. \tag{54}$$

We have to show that $a_n f/H_n \to a_0 f/H_0$ in $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, for any $f \in \boldsymbol{S}_0(\mathbb{R}^d)$. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ is Fourier invariant and $\mathcal{F}(a\text{⨿}_a) = \text{⨿}_\alpha$ with $\alpha = 1/a$ we have, recalling that $a_n \to a_0 \neq 0$, just to verify that

$$\lim_{n\to\infty} \|(\text{⨿}_{\alpha_n}\widehat{h}) * \widehat{f} - (\text{⨿}_{\alpha_0}\widehat{h}) * \widehat{f}\|_{\boldsymbol{S}_0} = 0. \tag{55}$$

In order to verify this result we invoke Theorem 5 of the recent paper [30] (or Theorem 2.2.ii) of [23]), showing that it is sufficient to verify that $\text{⨿}_{\alpha_n}\widehat{h}$ is a bounded family of measures in $\boldsymbol{M}_b(\mathbb{R}^d) = \boldsymbol{C}_0'(\mathbb{R}^d)$ which is w^*-convergent.

The boundedness is easy by direct verification using the sampling properties of $\boldsymbol{S}_0(\mathbb{R}^d)$, or by the observation that $\text{⨿}_{\alpha_n}\widehat{h}$ is a bounded sequence of translation-bounded measures (i.e. bounded in $\boldsymbol{W}(\boldsymbol{M}_b, \boldsymbol{\ell}^\infty)(\mathbb{R}^d)$) and thus by "coordinate-wise" multiplication (see e.g. Thm.2.1 in [31]):

$$\boldsymbol{W}(\boldsymbol{M}_b, \boldsymbol{\ell}^\infty) \cdot \boldsymbol{S}_0 \subset \boldsymbol{W}(\boldsymbol{M}_b, \boldsymbol{\ell}^\infty) \cdot \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1) \subset \boldsymbol{W}(\boldsymbol{M}_b, \boldsymbol{\ell}^1) = \boldsymbol{M}_b(\mathbb{R}^d)$$

provides an uniform estimate of the $\boldsymbol{M}_b$-norms of the products.

Due to the boundedness, it is sufficient to test the w^*-convergence for $\varphi \in \boldsymbol{C}_c(\mathbb{R}^d)$. The uniform continuity of φ, and the compactness of $\operatorname{supp}(\varphi)$ imply that the summation over $\mathbb{Z}^d$ can be restricted to a finite subset $F \subset \mathbb{Z}^d$:

$$|\text{⨿}_{\alpha_n}(\varphi) - \text{⨿}_{\alpha_0}(\varphi)| \leq \sum_{k\in F} |\varphi(k\alpha_n) - \varphi(k\alpha_0)| \to 0, \quad \text{for} \quad n \to \infty. \tag{56}$$

The combination of these two estimates is left to the reader in order to establish the claim of this proposition. □

Due to the Walnut representation, we can apply Proposition 5.1 for the diagonal preconditioner. For a Gabor atom $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ which is normalized in $\boldsymbol{L}^2(\mathbb{R}^d)$ the choice $h = |g|^2$ implies that one has $\widehat{h}(0) = \|g\|_2^{\,2} = 1$. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ is an algebra with respect to pointwise multiplication (and closed under conjugation) (see [24]) it is also clear that $h \in \boldsymbol{S}_0(\mathbb{R}^d)$ in this case. The assumptions of the above proposition are then applicable because we know that G_0 (arising in the Walnut representation) does not vanish in the case that (g, a_0, b_0) defines a Gabor frame. We thus have:

Corollary 5.3. *Assume that (g, a_0, b_0), with $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ defines a Gabor frame, then the conditions of Proposition 5.1 are satisfied. Hence the mapping $g \mapsto \mathfrak{D}_{g,\Lambda}(g)$ defines a continuous mapping, i.e. the diagonally preconditioned approximate dual atom depends continuously on g (in $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$) and the lattice parameters near a_0.*

Recall that the circulant preconditioners arise as diagonal preconditioners on the Fourier side:

$$\mathfrak{C}^{\mathcal{F}}_{g,\gamma,\Lambda} = \mathcal{F} \circ \mathfrak{D}_{\widehat{g},\widehat{\gamma},\Lambda'} \circ \mathcal{F}^* = \mathfrak{D}^{\mathcal{F}}_{\widehat{g},\widehat{\gamma},\Lambda'} \tag{57}$$

Thus we can derive the continuous dependence of the convolutionally preconditioned approximate dual Gabor atom, making use of the (isometric) Fourier invariance of $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$:

Corollary 5.4. *Assume that (g, a_0, b_0), with $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ defines a Gabor frame, the conditions of Prop. 5.1 hold for $h = |\widehat{g}|^2$. Hence the mapping $g \mapsto \mathfrak{C}_{g,\Lambda}(g)$ defines a continuous mapping in a neighborhood of b_0, i.e. the diagonally preconditioned approximate dual atom depends continuously on g (in $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$) and the lattice constant b.*

Our next goal is to combine these two observations, which turns out to the non-trivial, as it is not just a bilinear procedure. For the proof we need the following lemma providing a tightness statement for the collection of inverse elements with respect to convolution, for a family of tight measures. Let us first recall the concept of tightness (in the sense of uniform concentration) for bounded subsets of $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$[3].

[3]The word tightness has nothing to do with tightness in the sense of frames in Hilbert spaces!

Definition 5.5. *A bounded set of measures $S \subset (\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ is called tight if and only if $\forall \epsilon > 0$, $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that*

$$\|\mu - \mu \cdot k\|_{\boldsymbol{M}_b} \leq \varepsilon, \quad \forall \mu \in S.$$

Lemma 5.6. *Given a bounded, tight set S of bounded measures, inside a ball of radius $\gamma < 1$ in $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ we have the following situation:*

1. *For any $\nu \in S$ the geometric series of convolution powers $J(\nu) = \sum_{n=0}^{\infty} \nu^{*n}$ is absolutely convergent in $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$;*
2. *$J(S) = \{J(\nu), \nu \in S\}$ is a tight subset of $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$, with*

$$\|J(\nu)\|_{\boldsymbol{M}_b} \leq (1-\gamma)^{-1}, \quad \nu \in S.$$

Proof. First of all we observe that the series defining $J(\nu)$ is absolutely convergent in the Banach space $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$, with

$$\|J(\nu)\|_{\boldsymbol{M}_b} \leq \sum_{n=0}^{\infty} \|\nu^{*n}\|_{\boldsymbol{M}_b} \leq \sum_{n=0}^{\infty} \gamma^n \leq (1-\gamma)^{-1}.$$

Next, we recall that the convolution product of two tight subsets (in $\boldsymbol{\ell}^1(\alpha\mathbb{Z}^d)$, viewed as a subset of $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$) is again a tight subset (see [23], Lemma 2.1) hence - by induction - we gain (boundedness and) tightness of the sets $S^n := \{\nu^{*n}, \nu \in S\}$ for any $n \geq 1$, but also the addition of tight subsets is tight. This means that for every fixed $K \in \mathbb{N}$ the set of all partial sums to $J(\nu)$ of order K forms a tight and bounded subset. Noting that, by the choice of K, we can assure that one has for any $\nu \in S$:

$$\|J(\nu) - \sum_{n=0}^{K} \nu^{*n}\|_{\boldsymbol{M}_b} \leq \sum_{n=K+1}^{\infty} \gamma^n < \varepsilon/2.$$

Combining these to facts one finds that $J(S)$ itself will be a tight (and bounded) subset of $\boldsymbol{M}_b(\mathbb{R}^d)$. □

Based on these observations we can verify the continuous dependence of the approximate dual window obtained by the double preconditioning method. It is one of the analytic key results of this paper.

Theorem 5.7. *Double preconditioning depends continuously (in the strong sense by means of* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$*) on the parameters. In other words, given certain initial parameters* $g_0, \gamma_0 \in \boldsymbol{S}_0(\mathbb{R}^d)$*, some lattice* Λ_0 *and for a given* $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *and* $\varepsilon > 0$ *one has: Whenever* g, γ *are close to* g_0, γ_0 *in* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$*, and* Λ *close to* Λ_0 *in the sense of their generating* $2d \times 2d$ *- matrices, then one can assure that*

$$\|\mathfrak{P}_{g,\gamma,\Lambda}(f) - \mathfrak{P}_{g_0,\gamma_0,\Lambda_0}(f)\|_{\boldsymbol{S}_0} \leq \varepsilon. \tag{58}$$

This last estimate can be granted uniformly for $f \in M$*, where* M *is any compact subset in* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$*. An analog result is true for the alternative double preconditioner* $\mathfrak{R}_{g,\gamma,\Lambda}$*.*

Proof. Given a Gabor frame arising from (g, a, b), with $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, we know that the main diagonal of the Gabor frame operator is non-vanishing [43, Prop. 6.5.5.]. We thus can expect to apply Proposition 5.2 above, with $h = |g|^2 \in \boldsymbol{S}_0(\mathbb{R}^d)$ and $f = g$; The condition $\sum_{k\in\mathbb{Z}^d} |\widehat{h}_0(\alpha_0 k)| = 1 + \gamma' < 2$ is satisfied for all the lattice constants α_0 which are big enough. In fact, that $\widehat{h} = \mathcal{F}(|g|^2) = \widehat{g} * \widehat{g}^*$ belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$ ensures that the value is always finite and tends to $\widehat{h}(1) = \|g\|_2^2 = 1$ (by the usual normalization of Gabor atoms).

As the convolutional preconditioner is up to the conjugation with the Fourier transform (an automorphism of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$) just the diagonal preconditioner for $(\widehat{g}, b, a)$ the result for $\mathfrak{R}_{g,\gamma,\Lambda}$ follows by the same arguments. □

Theorem 5.8. *Assume that* (g_0, Λ_0) *defines a Gabor frame, for a given lattice* $\Lambda_0 = a_0\mathbb{Z}^d \times b_0\mathbb{Z}^d$ *and some Gabor atom* $g_0 \in \boldsymbol{S}_0(\mathbb{R}^d)$*. Given any compact set* $M \subset \boldsymbol{S}_0(\mathbb{R}^d)$ *containing* g_0 *and any sufficiently small compact neighborhood* U *of the lattice* Λ_0*, i.e. with* $|a - a_0| \leq \delta_0$ *and* $|b - b_0| \leq \delta_0$*.*

Then the family of operators $\mathfrak{P}_{g,\Lambda}$ *is uniformly bounded on* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ *(for* $g \in \boldsymbol{S}_0(\mathbb{R}^d)$*,* $\Lambda \in U$*), and for any fixed* $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *the mapping* $(g, \Lambda) \mapsto \mathfrak{P}_{g,\Lambda}(f)$ *is a continuous mapping from* $M \times U$ *into* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ *(with respect to the natural topologies).*

In particular, the double preconditioned approximate dual Gabor atom $\mathfrak{P}_{g,\Lambda}(g)$ *depends continuously on the pair* (g, Λ) *over* $M \times U$*.*

Proof. We have to show that for given (g_0, Λ_0) and $\varepsilon > 0$ we can find some neighborhood of (g_0, Λ_0) such that one has for all pairs of Gabor atoms g (close to g_0 in the sense of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$) and Λ (close to Λ_0):

$$I := \|\mathfrak{P}_{g_0,\Lambda_0}(g_0) - \mathfrak{P}_{g,\Lambda}(g)\|_{\boldsymbol{S}_0} \leq \varepsilon. \tag{59}$$

As a first step, we observe that Theorem 5.7 allows us to guarantee that all the preconditioners, in particular $\mathfrak{D}_{g,\Lambda}$, and $\mathfrak{C}_{\boldsymbol{A}}$ and $\mathfrak{P}_{g,\Lambda}$ are uniformly bounded on $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$. Let us denote the common bounds by C_0.

In order to obtain (59) we split the difference in the following way:

$$\mathfrak{P}_{g,\Lambda}(g) - \mathfrak{P}_{g_0,\Lambda_0}(g_0) = [\mathfrak{P}_{g,\Lambda}(g) - \mathfrak{P}_{g,\Lambda}(g_0)] + [\mathfrak{P}_{g,\Lambda}(g_0) - \mathfrak{P}_{g_0,\Lambda_0}(g_0)] \quad (60)$$

and so we have

$$[\mathfrak{P}_{g_0,\Lambda_0}(g_0) - \mathfrak{P}_{g,\Lambda}(g_0)] = [\mathfrak{C}_{\boldsymbol{A}_0} \circ (\mathfrak{D}_{g_0,\Lambda_0} - \mathfrak{D}_{g,\Lambda})(g_0)] + [\mathfrak{C}_{\boldsymbol{A}_0} - \mathfrak{C}_{\boldsymbol{A}}](\mathfrak{D}_{g,\Lambda}(g_0)). \quad (61)$$

Applying the triangular inequality gives us as a first estimate for (59):

$$I = \|\mathfrak{P}_{g_0,\Lambda_0}(g_0) - \mathfrak{P}_{g,\Lambda}(g)\|_{\boldsymbol{S}_0} \leq II + III + IV, \quad (62)$$

which are estimated separately. Thanks to the uniform bound on the preconditioners one has

$$II = \|\mathfrak{P}_{g,\Lambda}(g_0 - g)\|_{\boldsymbol{S}_0} \leq C_0^2 \, \|g_0 - g\|_{\boldsymbol{S}_0}, \quad (63)$$

$$III = \|\mathfrak{P}_{g_0,\Lambda_0}(g_0) - \mathfrak{P}_{g,\Lambda}(g_0)\|_{\boldsymbol{S}_0} = \|[\mathfrak{C}_{\boldsymbol{A}_0} \circ \mathfrak{D}_{g_0,\Lambda_0}](g_0) - [\mathfrak{C}_{\boldsymbol{A}_0} \circ \mathfrak{D}_{g,\Lambda}](g_0)\|_{\boldsymbol{S}_0}, \quad (64)$$

and finally

$$IV = \|[\mathfrak{C}_{\boldsymbol{A}_0} - \mathfrak{C}_{\boldsymbol{A}}](\mathfrak{D}_{g,\Lambda}(g_0))\|_{\boldsymbol{S}_0}. \quad (65)$$

Let us treat each of the terms separately, and ensure that the overall difference for (62) can be estimated by $\varepsilon > 0$ by making sure that each of the three estimates allows coming up with an estimate of size $\varepsilon/4$.

Continuing the estimate (64) for III above we have

$$III \leq |||\mathfrak{C}_{\boldsymbol{A}_0}|||_{\boldsymbol{S}_0} \cdot \|\mathfrak{D}_{g_0,\Lambda_0}(g_0) - \mathfrak{D}_{g_0,\Lambda}(g_0)\|_{\boldsymbol{S}_0} \leq C_0 \|\mathfrak{D}_{g_0,\Lambda_0}(g_0) - \mathfrak{D}_{g,\Lambda}(g_0)\|_{\boldsymbol{S}_0}. \quad (66)$$

This term can be controlled by invoking (53) and choosing the neighborhoods of g_0 and Λ_0 small enough to ensure, for fixed $f_0 = g_0$ we have

$$\|\mathfrak{D}_{g_0,\Lambda_0}(g_0) - \mathfrak{D}_{g,\Lambda}(g_0)\|_{\boldsymbol{S}_0} \leq \varepsilon/(4C_0). \quad (67)$$

The most delicate part is the final estimate for IV. Here it is not enough to observe that the considered set $\{\mathfrak{D}_{g,\Lambda}(g_0)\}$ is bounded, because we cannot expect a norm estimate for the difference operator $[\mathfrak{C}_{\boldsymbol{A}_0} - \mathfrak{C}_{\boldsymbol{A}}]$ to hold true, as we only have strong convergence combined with uniform boundedness.

However, we can save the situation, because the explicit description of the set $\{\mathfrak{D}_{g,\Lambda}(g_0)\} \subset \boldsymbol{S}_0(\mathbb{R}^d)$ allows us to invoke the general compactness criterion given in [25] or [28] for the Banach space $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$. As a matter of fact, the admitted diagonal preconditioners define a family of functions which is *equicontinuous* in the Fourier algebra, because it required (up to a small error) only the multiplication with pure frequencies up to some given maximal frequency. Multiplying a single function (here the Gabor atom g) with such a family of diagonally preconditioned windows $\mathfrak{D}_{g,\Lambda}(g_0)$ which form a bounded, tight and equicontinuous, hence a relatively compact subset of $\boldsymbol{S}_0(\mathbb{R}^d)$.

The fact that the pointwise product of two bounded, equicontinuous families is again equicontinuous, but now also tight if one of them is tight, is discussed in the Lemma presented in in Appendix 2. As a consequence one obtains relative compactness in $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$. □

Remark 5.9. *The continuous dependence provided above implies among others that the exact choice of the local lattice constants for the double preconditioners is of minor importance because small changes will not have a significant impact on the computed approximate dual Gabor atom. This is particularly useful to have in mind for slowly varying lattice constants, as it is discussed in the quasi-regular situation (see section 8.3 below).*

6. Double Preconditioning for the Banach Gelfand Triple

In the last section, we have shown that double preconditioning makes also sense in the continuous case. Now we want to explore the potential of this method in the context of the Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ (see Section 2.1.3).

In the most general setting, the starting point is the observation that the Gabor frame operator generated by the pair (g, Λ) commutes with all the TF-shifts from the lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. This implies that the Gabor frame operator has a spreading symbol which is supported on the *adjoint lattice* $\Lambda^\circ \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Even in the most general setting, meaning that we only have $g \in \boldsymbol{L}^2(\mathbb{R}^d)$, without the usual Bessel condition on the Gabor family (g, Λ), we know that the frame operator $\mathbf{S}_{g,\Lambda}$ - which possibly is unbounded on $\boldsymbol{L}^2(\mathbb{R}^d)$ - defines a bounded linear operator from $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ to $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$. By a combination of Thm.7.7.5 in [35] with Thm.3.3.1 of [41] this implies a spreading representation of the form

$$\mathbf{S}_{g,\Lambda} = C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} V_g g(\lambda^\circ) \pi(\lambda^\circ), \tag{68}$$

with coefficients which are a multiple of $(V_g g(\lambda^\circ))_{\lambda^\circ \in \Lambda^\circ}$ in $\boldsymbol{c}_0(\Lambda^\circ)$, the space of bounded sequences over Λ°, tending to zero at infinity. To see this, note that $V_g g \in \boldsymbol{C}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ for $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ and $V_g g \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ by [41]. On the other hand $\|V_g g\|_\infty \leq \|g\|_2^2$ for $g \in \boldsymbol{L}^2(\mathbb{R}^d)$, hence $V_g g$ belongs to the closure of $\boldsymbol{S}_0(\mathbb{R}^{2d})$ in $\boldsymbol{L}^\infty(\mathbb{R}^{2d})$ with respect to the sup-norm, i.e. to $\boldsymbol{C}_0(\mathbb{R}^{2d})$. For all practical applications, it is very reasonable to assume at least that $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ (see [43]). In such a case it is well known that $V_g g \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and hence these coefficients belong to $\boldsymbol{\ell}^1(\Lambda^\circ)$, with the corresponding estimate:

$$\|(V_g g(\lambda^\circ))_{\lambda^\circ \in \Lambda^\circ}\|_{\boldsymbol{\ell}^1(\Lambda^\circ)} \leq C_\Lambda^1 \|g\|_{\boldsymbol{S}_0}{}^2, \quad g \in \boldsymbol{S}_0(\mathbb{R}^d). \tag{69}$$

The reader may consult Section 13.4 of [43] for considerations concerning the *rotation algebra* which describes the composition of operators with such a concentration of the spreading function via *twisted convolution* (see [43]). The problem of determining, whether an operator in such a non-commutative Banach algebra is invertible, and in fact the wish to approximate this inverse in an efficient way, is thus a non-trivial mathematical task.

The approach taken in [44] makes use of spectral methods for non-commutative Banach algebras and shows the interesting (and deep) fact that for Gabor atoms $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ the invertibility of the frame operator $\mathbf{S}_{g,\Lambda}$ on the Hilbert space implies that its restriction to $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ is also defining an invertible automorphism for $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$. The boundedness of $\mathbf{S}_{g,\Lambda}$ on $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ is not very difficult (see [41]), the point is the (abstract claim of) bounded invertibility on $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$, which is equivalent to the condition that the dual atom $\widetilde{g}$ also belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$. In fact, we have $\widetilde{g} := \mathbf{S}_{g,\Lambda}^{-1}(g)$, but since the inverse of the frame operator $\mathbf{S}_{g,\Lambda}$ is just $\mathbf{S}_{\widetilde{g},\Lambda}$ the converse is true as well. This gives the starting point for the theory of modulation spaces [27, 22], and localized frames [42, 5].

However, one must say, that the abstract methods of the papers by Gröchenig and Leinert ([44], for the rational case see also [32]) do not provide explicit norm estimates for $\widetilde{g}$ in $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ respectively of the operator norm of the inverse frame operator on $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$.

There are certain cases, where the invertibility of the Gabor frame operator is easy because the frame operator takes a particularly simple form. For these cases, it is also easy to compute the square-root inverse of the frame operator, respectively the canonical tight Gabor atom. Let us list such cases:

1. It is possible that the frame operator is just a multiplication operator. This can be seen in the typical case of a triple (g, a, b), with a compactly

supported Gabor atom g with $\mathrm{supp}(g) \subseteq B_\delta(0)$ for some $\delta \leq 1/2b$. This *painless case* has been described already in one of the first papers on Gabor analysis (see [17]). An equivalent statement is also possible for non-stationary Gabor transforms with the same - but time-varying - support sufficient condition [3].
More generally, it is enough that the diameter of the support of g is at most $1/b$, because the property of generating a Gabor frame is invariant under translation of the Gabor atom, i.e. it is valid for g if and only if it is true for some or any translate $T_y g$. In fact, the Gabor frame operator for $T_y g$ is unitary equivalent to $\mathbf{S}_{g,\Lambda}$ by conjugation with the shift operator T_y, i.e.

$$\mathbf{S}_{T_y g,\Lambda} = T_y \circ \mathbf{S}_{g,\Lambda} \circ T_y', \quad y \in \mathbb{R}^d.$$

In a similar way one can replace g by $\pi(\lambda)g$, i.e. we have

$$\mathbf{S}_{\pi(\lambda)g,\Lambda} = \pi(\lambda) \circ \mathbf{S}_{g,\Lambda} \circ \pi(\lambda)', \quad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d. \tag{70}$$

We will discuss an extension of this observation in subsection 6.2.

2. Using the observation that the Gabor frame operator is (via the Fourier transform) equivalent to the Gabor frame operator $(S_{\widehat{g}}, b, a)$ we find that the Gabor frame operator allows a *painless inversion* on the Fourier transform side, if g is a band-limited function with $\mathrm{supp}(\widehat{g}) \subset B_\gamma(0)$, for some $\gamma < 1/2a$.
3. Finally, we should discuss the case of a critical lattice, which is in fact (at least for all practical purposes) a lattice of the form $\Lambda = \Lambda_1 \times \Lambda_1^\perp$. In this case the TF-shifts from Λ form a commutative group, i.e. the mapping $\lambda \mapsto \pi(\lambda)$ is not just a *projective representation* (see [11], or [43]) but it is in fact a true, unitary representation. This means that the composition laws for this family of TF-shifts is isomorphic to the corresponding group of shift operators acting on Λ. Since Λ is a commutative group under addition we are faced with the simple problem of inverting a convolution operator (acting on Λ), which is easy, making use of Fourier transform methods. However, in the concrete case of the standard *von Neumann lattice* $\Lambda = \mathbb{Z} \times \mathbb{Z} \lhd \mathbb{R} \times \widehat{\mathbb{R}}$ one better refers to the (equivalent, but much better known) *Zak transform*, which is known to provide a highly efficient method for the treatment of TF-lattices with integer redundancy. For the critical case, this method is not very useful in practice and with respect to our discussion, because

the *Balian-Low principle* prohibits the existence of atoms $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ such that $(g, a, \frac{1}{a})$ generates a Gabor frame for $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$ (see [43]).

One can say, that the idea of *double preconditioning* is essentially based on the following observation: If the time-localization of g is not satisfied exactly, but at least approximately, then the inversion of the diagonal part of the matrix is expected to provide a good approximate inversion. If the same happens on Fourier side then the circulant part of the Gabor frame operators will improve the situation. In certain cases one can expect that their combination, done in the right way, may even perform better.

Back to the diagonal preconditioner on can say that such considerations have been already pursued in an early paper by Casazza and Christensen, formulating sufficient conditions for the invertibility of the Gabor frame operator on $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$, see [9] (Thm.2.1). Their approach was extending earlier results by I. Daubechies ([16], Thm.2.5). In the finite discrete case, it corresponds essentially to the diagonal dominance of the corresponding matrix (which always implies invertibility).

6.1. Banach Algebra Generated by Adjoint TF-shifts

For the functional analytic description the following Banach algebra will be of relevance. Given a lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ (i.e. $\Lambda = \boldsymbol{A} * \mathbb{Z}^{2d}$ for some non-singular real $2d \times 2d$-matrix $\boldsymbol{A}$) we will consider the algebra

$$\mathcal{A}^{\Lambda^\circ} := \{T = \sum_{\lambda^\circ \in \Lambda^\circ} c_{\lambda^\circ} \pi(\lambda^\circ) \text{ with } \sum_{\lambda^\circ \in \Lambda^\circ} |c_{\lambda^\circ}| < \infty\}. \tag{71}$$

Here Λ° is the adjoint lattice, which turns out to be - up to an exchange of coordinates - to be given by $(\boldsymbol{A}^1)^{-1} * \mathbb{Z}^{2d}$ (see [41], Chap. 3.5.3).

For the most interesting cases, i.e. for lattice Λ with modest redundancy, this representation is in fact unique (because the operators $\{\pi(\lambda^\circ), \lambda^\circ \in \Lambda^\circ\}$ form a Riesz projection basis for $\mathcal{A}^{\Lambda^\circ}$, see [29] and [41] for the terminology). However, to make the discussion easier let us take the naive norm for this space, which is the infimum over the $\boldsymbol{\ell}^1$-norm of coefficients over all admissible representations of T as in (71) as follows:

$$\|T \,|\, \mathcal{A}^{\Lambda^\circ}\| = \|T\|_{\mathcal{A}^{\Lambda^\circ}} := \inf_{admiss.repr.} \sum_{\lambda^\circ \in \Lambda^\circ} |c_{\lambda^\circ}|, \tag{72}$$

where the Infinium is taken over all "admissible representations" of T, i.e. a representation as in (71). Let us collect a few facts about this algebra (which has appeared in several papers on Gabor analysis, see [47], for example):

Theorem 6.1. *Given any $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ we have:*

1. $\left(\mathcal{A}^{\Lambda^\circ}, \|\cdot|\mathcal{A}^{\Lambda^\circ}\|\right)$ *is a self-adjoint Banach algebra under composition.*
2. $\mathcal{A}^{\Lambda^\circ}$ *is a subalgebra of all the operators which are Banach Gelfand triple morphism for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, commuting with all the TF-shifts arising from $\{\pi(\lambda), \lambda \in \Lambda\}$.*
3. *For any $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ such that (g, Λ) defines a Gabor frame for $\boldsymbol{L}^2(\mathbb{R}^d)$ the Gabor frame operator defines an invertible element in $\mathcal{A}^{\Lambda^\circ}$.*

Remark 6.2. *Depending on the structure of Λ resp. Λ° the algebra $\mathcal{A}^{\Lambda^\circ}$ will be commutative or not. In the general case, the composition can be described in a unique way via twisted convolution, but the point of our discussion is, that we do not have to go deeper into this form of the analysis in order to appreciate the role of double preconditioning.*

Since Theorem 6.1 is not stated in the required form in the published literature let us provide a proof for the convenience of the reader.

Proof. i) The norm property and the completeness are routine questions that are left to the interested reader. The fact that it is a Banach algebra under composition (which can be described as twisted convolution, see [43]) is based on the combination of two facts:

- The twisted convolution is coordinate-wise dominated by the ordinary convolution (of two sequences on the Abelian group Λ°, with addition).
- $\left(\boldsymbol{\ell}^1(\Lambda^\circ), \|\cdot\|_{\boldsymbol{\ell}^1(\Lambda^\circ)}\right)$ is a solid Banach algebra with respect to convolution (over Λ°), i.e. it is closed under convolution, and furthermore an estimate $|a_{\lambda^\circ}| \leq |b_{\lambda^\circ}|$ for two sequences over Λ°, with $b \in \boldsymbol{\ell}^1(\Lambda^\circ)$, implies that also $a \in \boldsymbol{\ell}^1(\Lambda^\circ)$ and $\|a\|_{\boldsymbol{\ell}^1} \leq \|b\|_{\boldsymbol{\ell}^1}$.

ii) Observe that $\mathcal{A}^{\Lambda^\circ}$ is continuously embedded into the operator algebra of any Banach space of functions or distributions for which TF-shifts act isometrically, i.e. with

$$\|\pi(\lambda)(f)\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \quad f \in \boldsymbol{B}. \tag{73}$$

In fact, the norm in $\left(\mathcal{A}^{\Lambda^\circ}, \|\cdot|\mathcal{A}^{\Lambda^\circ}\|\right)$ dominates the operator norm on any of these spaces, since we have for any $T = \sum_{\lambda^\circ\in\Lambda^\circ} c_{\lambda^\circ}\pi(\lambda^\circ) \in \mathcal{A}^{\Lambda^\circ}$ the obvious estimate

$$\|f\|_{\boldsymbol{B}} \leq \sum_{\lambda^\circ\in\Lambda^\circ} |c_{\lambda^\circ}| \|\pi(\lambda^\circ)(f)\|_{\boldsymbol{B}} \leq \|f\|_{\boldsymbol{B}} \sum_{\lambda^\circ\in\Lambda^\circ} |c_{\lambda^\circ}|. \tag{74}$$

Since this estimate is valid for any representation one can take the infimum over the right-hand side and obtain in this way

$$\|T(f)\|_{\boldsymbol{B}} \leq \|f\|_{\boldsymbol{B}} \|T\,|\,\mathcal{A}^{\Lambda^\circ}\|, \quad f \in \boldsymbol{B}. \tag{75}$$

In other words, $\left(\mathcal{A}^{\Lambda^\circ}, \|\cdot|\mathcal{A}^{\Lambda^\circ}\|\right)$ is continuously embedded into the operator algebra over such a space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Of course, the three spaces constituting the BGT $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ are in this category. The w^*-w^*--continuity is also easily verified, since the adjoint operator (in the Hilbert space sense) also belongs to $\mathcal{A}^{\Lambda^\circ}$ (with the same norm), and $T = T^{**}$.

iii) We know from Janssen's representation (12) that the Gabor frame operator $\mathbf{S}_{g,\Lambda}$ is of the required form, with coefficients arising as samples of $V_g g$ over the *adjoint lattice*, i.e. a multiple of $(\mathrm{red}(\Lambda)V_g g(\lambda^\circ))_{\lambda^\circ\in\Lambda^\circ}$. Since we know that $V_g g \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ the membership of $\mathbf{S}_{g,\Lambda}$ in $\mathcal{A}^{\Lambda^\circ}$ can be granted (see the estimate (69) above). □

In this setting the so-called Janssen Criterion comes in easily:

Lemma 6.3. *Any operator $T \in \mathcal{A}^{\Lambda^\circ}$, such that for some $\gamma < 1$ one has*

$$\sum_{0\neq\lambda^\circ\in\Lambda^\circ} |c_{\lambda^\circ}| < \gamma\,|c_0| \tag{76}$$

with $\|T^{-1}\,|\,\mathcal{A}^{\Lambda^\circ}\| \leq ((1-\gamma)|c_0|)^{-1}$.

Proof. The assumptions imply that $c_0 \neq 0$, so we may divide by it. Since we have $Id = \pi(0)$ (with $0 = (0,0) \in \Lambda^\circ$), the multiplicative identity in the Banach algebra $\mathcal{A}^{\Lambda^\circ}$, condition (76) can be expressed as the estimate

$$\|Id - \frac{1}{c_0}T\|_{\mathcal{A}^{\Lambda^\circ}} < \gamma < 1,$$

which, by standard arguments of Banach algebra theory (using a Neumann series) guarantees the invertibility of T and the norm estimate via a geometric series giving $\|\left[\frac{1}{c_0}\cdot T\right]^{-1}\,|\,\mathcal{A}^{\Lambda^\circ}\| = |c_0|\cdot\|T^{-1}\,|\,\mathcal{A}^{\Lambda^\circ}\| < (1-\gamma)^{-1}$. □

The goal of preconditioning is, of course, to bring a given operator closer to the identity operator. The standard idea is to choose some approximations of the given operator which is easily invertible and do a good job in this sense. We formulate this principle in the following way:

Corollary 6.4. *Assume that we have* $\|\mathrm{Id} - B \circ T \,|\, \mathcal{A}^{\Lambda^\circ}\| \leq \gamma < 1$*, for some invertible operator* $B \in \mathcal{A}^{\Lambda^\circ}$*. Then* T *is invertible in* $\mathcal{A}^{\Lambda^\circ}$ *and we have*

$$\|T^{-1} \,|\, \mathcal{A}^{\Lambda^\circ}\| \leq (1-\gamma)^{-1} \|B^{-1} \,|\, \mathcal{A}^{\Lambda^\circ}\|. \tag{77}$$

As the numerical experiments indicate the natural choices of such simple preconditioners from the Banach algebra $\mathcal{A}^{\Lambda^\circ}$, namely the central multiplier of the *Walnut representation* (respectively the diagonal part of the Gabor frame matrix, in the finite-dimensional case) and the analog on the Fourier transform side turns out to work quite efficiently and enables in most cases of interest to establish the invertibility of the Gabor frame operator.

In fact, for the classic situation of a Gabor frame with the input (g, a, b) the adjoint lattice is just $(1/b)\mathbb{Z} \times (1/a)\mathbb{Z}$. Thus the Janssen criterion requires a normalized atom with $\|g\|_2 = 1$ that

$$\sum_{(k,n) \in \mathbb{Z}^d \times \mathbb{Z}^d} |V_g g(k/b, n/a)| = 1 + \gamma < 2, \tag{78}$$

while for the invertibility of each of the two preconditioners only one of the following two estimates is required:

$$\sum_{n \in \mathbb{Z}^d} |V_g g(0, n/a)| < 1 + \gamma_1, \tag{79}$$

respectively for the Fourier version:

$$\sum_{k \in \mathbb{Z}^d} |V_g g(k/b, 0)| < 1 + \gamma_2, \tag{80}$$

for suitable values $\gamma, \gamma_1, \gamma_2 < 1$. We leave the optimization of parameters to the interested reader, as it will depend on the configuration (the lattice, the window, and so on).

6.2. Benefit of the Choice of Preconditioners

We have seen that the preconditioners used so far all have in common that they arise as operators in the Banach algebra $\mathcal{A}^{\Lambda^\circ}$, and thus act uniformly on all the Banach spaces $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ on which TF-shifts $\pi(\lambda)$ act isometrically.

As a benefit, we can point out two possible motivations for the use of preconditioners can be put into a logical connection using this common fact. Because the results given in this subsection apply to any of the preconditioners discussed so far we use the generic symbol $\mathfrak{T}$ for a typical operator of this kind, noting that they are all special cases of operators as in the previous section.

The inversion problem for the Gabor frame operator $\mathbf{S}_{g,\Lambda}$ and the construction of the *canonical dual Gabor atom* $\widetilde{g} = \mathbf{S}_{g,\Lambda}^{-1}(g)$ are equivalent tasks, due to the well-known relation $\mathbf{S}_{g,\Lambda}^{-1} = \mathbf{S}_{\widetilde{g},\Lambda}$. For similar reasons the following two aspects of Gabor theory are closely related:

1. Given a (re-scaled version of) $\mathbf{S}_{g,\Lambda}$, which might be too far away from the identity operator, one might want to find a suitable preconditioner $\mathfrak{T}$ such that $\mathfrak{T} \circ \mathbf{S}_{g,\Lambda}$ is getting closer to Id (as operator on $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$ or on the BGT);
2. Starting from the initial guess h for a dual atom, we want to find a better approximation of $\widetilde{g}$ by applying some operator $\mathfrak{T}$ to h, and then use $\mathfrak{T}h$ as an improved approximation to $\widetilde{g}$.

We show that in fact, these two perspectives are equivalent!

Lemma 6.5. *For any $\mathfrak{T} \in \mathcal{A}^{\Lambda^\circ}$ one has*

$$\mathfrak{T} \circ \mathbf{S}_{g,h,\Lambda} = \mathbf{S}_{g,\mathfrak{T}h,\Lambda}, \tag{81}$$

and

$$\mathbf{S}_{g,h,\Lambda} \circ \mathfrak{T} = \mathbf{S}_{\mathfrak{T}^* g,h,\Lambda} \tag{82}$$

Proof. Let us provide a detailed proof for (82), formula (81) can be verified by similar arguments.

By the definition of Λ° we have

$$\pi(\lambda) \circ \pi(\lambda^\circ) = \pi(\lambda^\circ) \circ \pi(\lambda), \quad \forall \lambda \in \Lambda,\, \lambda^\circ \in \Lambda^\circ.$$

Since any $\mathfrak{T} \in \mathcal{A}^{\Lambda^\circ}$ (or also $\mathfrak{T}^* \in \mathcal{A}^{\Lambda^\circ}$) can be approximated by finite linear combinations of operators $\pi(\lambda^\circ), \lambda^\circ \in \Lambda^\circ$, it is clear that we have

$$\pi(\lambda) \circ \mathfrak{T} = \mathfrak{T} \circ \pi(\lambda), \quad \lambda \in \Lambda. \tag{83}$$

Since $\mathfrak{T} \in \mathcal{A}^{\Lambda^\circ}$ if and only if $\mathfrak{T}^* \in \mathcal{A}^{\Lambda^\circ}$ a similar commutation relation holds true for $\mathfrak{T}^*$.

Next, we use that frame-like operators have an absolutely convergent Janssen representation for any pair $g, h \in \boldsymbol{S}_0(\mathbb{R}^d)$, i.e.

$$\mathbf{S}_{g,h,\Lambda}(\mathfrak{T}f) = \sum_{\lambda\in\Lambda} \langle \mathfrak{T}f, \pi(\lambda)g\rangle \pi(\lambda)\, h = \sum_{\lambda\in\Lambda} \langle f, \mathfrak{T}^*(\pi(\lambda)g)\rangle\, \pi(\lambda)\, h = \tag{84}$$

$$= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)(\mathfrak{T}^* g)\rangle \pi(\lambda)\, h = \mathbf{S}_{\mathfrak{T}^* g,h,\Lambda} f, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \tag{85}$$

□

In our applications the different preconditioners are self-adjoint (in fact strictly positive) operators, and thus in such a case $\mathfrak{T}^* = \mathfrak{T}$ and we can see that the (according to our numerical experience, most of the time) excellent approximation of $\widetilde{g}$ by $\mathfrak{T}(h)$ allows the estimate

$$\|f - \mathbf{S}_{g,\mathfrak{T}h,\Lambda}(f)\|_{\boldsymbol{B}} \le \|[\mathbf{S}_{g,\widetilde{g},\Lambda} - \mathbf{S}_{g,\mathfrak{T}h,\Lambda}](f)\|_{\boldsymbol{B}} \le C_\Lambda \cdot \|\widetilde{g} - \mathfrak{T}h\|_{\boldsymbol{S}_0} \|g\|_{\boldsymbol{S}_0} \|f\|_{\boldsymbol{B}}. \tag{86}$$

Equivalently, we have in this situation

$$\|\|\mathrm{Id} - \mathbf{S}_{g,\mathfrak{T}h,\Lambda}\|\|_{\boldsymbol{B}\to\boldsymbol{B}} \le C_\Lambda \cdot \|g\|_{\boldsymbol{S}_0} \cdot \|\widetilde{g} - \mathfrak{T}h\|_{\boldsymbol{S}_0}. \tag{87}$$

This shows that $\mathfrak{T}h$ provides a good approximate reconstruction from the sampled STFT of f with window g, with control of the reconstruction error (in the operator norm on $\big(\boldsymbol{L}^2(\mathbb{R}^d),\, \|\cdot\|_2\big)$ or $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$, for example).

The other version makes use of the approximate dual window for synthesis and then one can say, that $\mathfrak{T}h$ provides a good approximate recovery of f from the samples $(V_g f(\lambda))_{\lambda\in\Lambda}$:

$$\|f - \mathbf{S}_{\mathfrak{T}h,g,\Lambda}(f)\|_{\boldsymbol{B}} \le \|\mathbf{S}_{\widetilde{g},g,\Lambda} - \mathbf{S}_{\mathfrak{T}h,g,\Lambda}(f)\|_{\boldsymbol{B}} \le C_\Lambda \cdot \|\widetilde{g} - \mathfrak{T}h\|_{\boldsymbol{S}_0} \|g\|_{\boldsymbol{S}_0} \|f\|_{\boldsymbol{B}}. \tag{88}$$

Again, this is valid for $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}}) = \big(\boldsymbol{L}^2(\mathbb{R}^d),\, \|\cdot\|_2\big)$ or $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$.

Note that the constant C_Λ does not depend on the particular choice of $\boldsymbol{B}$ as long as the TF-shifts $\pi(\lambda)$ act isometrically on $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$.

7. Numerical Exposition

In this section, we provide numerical demonstrations that show the quality of the preconditioning method for different lattices, hence redundancy

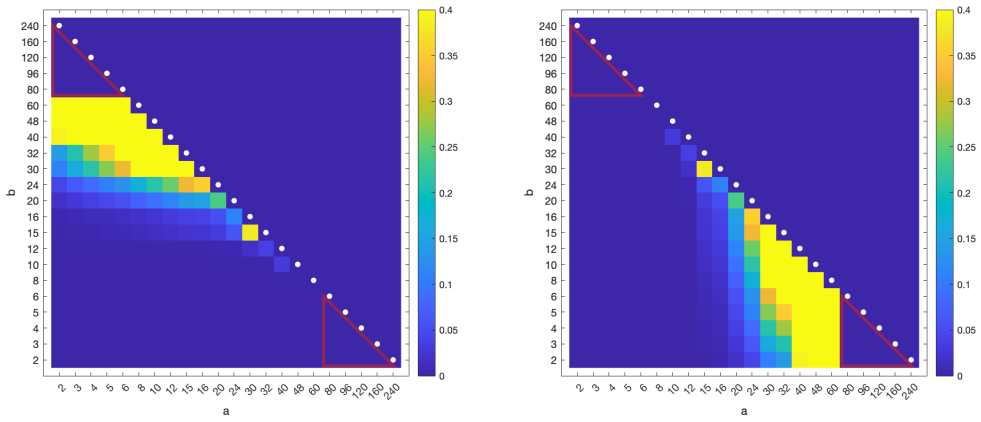

Figure 3: Left: L^2 distances of diagonal preconditioned duals to the canonical duals. Right: L^2 distance of preconditioned duals to the canonical duals.

settings. The starting point for the computations is a periodized Gaussian as window function with a length of $n = 480$ samples. For other signal lengths (preferably with many divisors) the results are similar. In other words, we present the results for a *typical* case.

For n, all divisors are computed, so that all possible choices of regular lattices $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ can be considered. The different hop sizes are aligned to the axes: the ones corresponding to time, denoted by a to the x-axis, and the ones for frequency, b to the y-axis, both in an increasing manner. This produces table plots where high redundancy systems pertain to the lower left corner and low redundancies to the upper right one. The diagonal (white dots in the figures) indicate the cases where the redundancy of the Gabor system is critical, i.e. $\frac{n}{a \cdot b} = 1$. For each pair below this diagonal, approximate canonical dual windows are computed and compared to the canonical duals. Very eccentric lattices, i.e. where $\frac{a}{b}$ or $\frac{b}{a}$ is large yield ill-conditioned Gabor systems and thus, are not computed in the plots. The corresponding cases are indicated by the red triangle-shaped framing.

7.1. Single Preconditioning

As previously discussed, considering the single preconditioners in the context of the Janssen representation yields the following expectation:

1. Diagonal preconditioning yields good approximate duals for small b

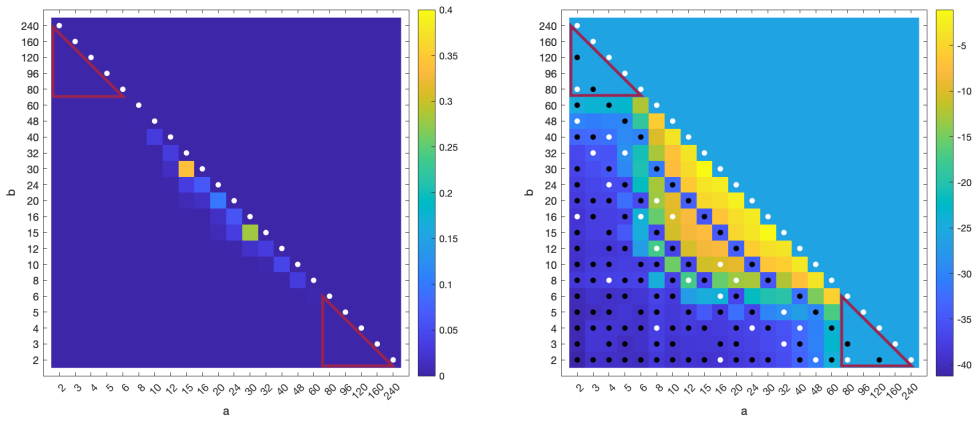

Figure 4: Left: L^2 distances of double preconditioned dual windows to canonical dual windows across all (non-eccentric) lattices. Right: Logarithmic L^2 distances.

2. Circulant preconditioning yields good approximate duals for small a

Thinking of the structure of the Gabor frame matrix, this corresponds to the intuition that the approximations should be good when the matrix is diagonally dominant, either directly or in the Fourier domain. Figure 3 shows the above-described table plot with logarithmic error values for both single preconditioning methods applied separately, showing the expected behaviors.

7.2. Double Preconditioning

Due to its recursive manner, the double preconditioner can benefit from both single variants, which can be well observed in Figure 4. One immediately notices that the method works impressively well, even for redundancies close to the critical border. Having in mind that, in fact, a decent amount of redundancy is required for many applications, the method already yields very good approximations for most situations of practical interest. One further notices that the error values are symmetric in a and b, which is a consequence of the Gaussian window with a time-frequency ratio of 1. When dilating the window, a quality shift occurs, according to what one would expect: For wide windows, the error gets worse for large values of b and vice versa for narrow windows.

Additionally, we observe an interesting behavior for the cases where the redundancies are *even integers*. The right plot in Figure 4 emphasizes this

pattern by showing $\log(\|\tilde{g}^{(ap)} - \tilde{g}\|)$. Black dots mark the even redundancy cases and white dots the odd redundancy cases. To double-check this phenomenon, we verified that this pattern does not occur in the case where n is given by a power of two. For the moment we can only conjecture why this is happening and postpone a more detailed discussion to future work.

8. Application

8.1. Approximation of the Canonical Tight Gabor Atom

A (Gabor) frame is called *tight* if the optimal frame bounds A, B coincide. As a consequence, the corresponding frame operator is just a multiple of the identity operator, i.e. $\mathbf{S}_{g,\Lambda} = A \cdot I$, so that by a simple re-scaling (by the factor $1/\sqrt{A}$) one obtains a so-called *Parseval* frame with $A = B = 1$. Obviously, the canonical dual frame coincides with the original frame in such a case.

Such Gabor frames are particularly useful to the implementation of Gabor multipliers because one can expect that signal concentrated in areas where the multiplier is close to a given numerical value will be more or less just multiplied (enhanced or damped) by this very factor. Tight frames have the additional advantage that real-valued multipliers induce self-adjoint operators, and the constant symbol 1 gives the identity (for details in a more general context see [2]), as one may expect from a *symbolic calculus*. Furthermore, if the multiplier (upper) symbol takes values in the interval $[\alpha, \beta]$ for $0 < \alpha \leq \beta < \infty$ then the eigenvalues of such a symmetric operator will be inside this same interval (see [18] and [39]).

The usual way to generate a (canonical) tight Gabor frame in our situation is to apply the inverse square root of the (positive definite) frame operator to g, i.e. to obtain a new generator $g^t = \mathbf{S}_{g,\Lambda}^{-1/2} g$, which is known to be among all the possible candidates in $\boldsymbol{L}^2(\mathbb{R}^d)$ the one which is most close to the original choice $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ while generating a tight Gabor frame for the lattice Λ.

One can extend the double preconditioning method in a straightforward manner to approximate g^t and since computing the square root of a multiplication operator is easy,

$$\mathfrak{P}^t = \mathfrak{C}^{1/2}\big(\mathfrak{D}^{1/2}\mathbf{S}_{g,\Lambda}\big)\mathfrak{D}^{1/2} \tag{89}$$

provides an efficient and good approximation to g^t.

We shall illustrate also numerically that this approximation scheme works in general impressively well. The approximation quality, clearly, is comparable to the original application of approximating the canonical dual. Figure

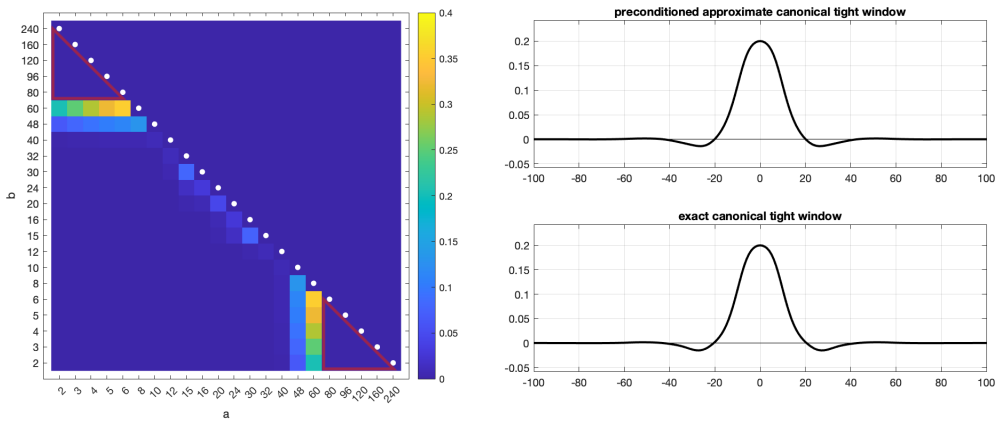

Figure 5: Left: L^2 distances of double preconditioned canonical tight windows to canonical tight windows across all (non-eccentric) lattices. Right: Window comparison for the lattice where $a = 16$ and $b = 20$, i.e. the Gabor system has a redundancy of 1.5.

5 shows the initially described table plot for all possible (yet reasonable) lattices and a direct comparison of the approximated tight and the original tight window.

8.1.1. The FAB method

As mentioned above tight windows have big advantages, so there are several approaches to construct tight Gabor frames [20], in particular, for the application of orthogonal frequency division multiplexing (OFDM) [7]. The method presented there, called *FAB-method*, is described as follows: For fixed $a = \frac{1}{2}$ and $b = 1$ and $g(t) = (2\alpha)^{1/4} e^{-\pi\alpha t^2}$, it is defined as

$$h^{(FAB)} = O_a F^* O_b F(g), \tag{90}$$

where

$$O_\lambda g = \left(\lambda \sum_{i=-\infty}^{\infty} |g(t - k\lambda)|^2 \right)^{-1/2} (g). \tag{91}$$

Clearly, this corresponds precisely (up to a scaling factor) to the double preconditioning procedure to compute an approximate canonical tight Gabor atom, where the circulant preconditioner is applied first. Hence in our notation, we can extend the FAB-method to any a, b and g by defining

$$g^{(FAB)} = \mathfrak{D}^{1/2}_{g,g^{(t,C)},\Lambda} \mathfrak{C}^{1/2}_{g,\Lambda}(g), \tag{92}$$

where $g^{(t,C)} = \mathfrak{C}_{g,\Lambda}^{1/2}(g)$. Note that since the goal here is to produce a tight Gabor frame, the scaling in (91) is not crucial.

8.2. Iterative Procedure

It was shown that for decent redundancy and "nice" windows, a double preconditioned window already yields a very good approximation of the canonical dual window and thus, is certainly sufficient for many applications. However, as already mentioned in [4] double preconditioning provides a very good starting point for the Neumann algorithm, i.e. using PS instead of S. We would like to mention a simple trick to instantly improve the algorithm, based on the Euler method, or *power of two trick*. For this, note that using P as preconditioner, $\|I - PS\| < 1$ holds and thus, clearly also $\|(I - PS)^2\| < \|I - PS\|$. In the matrix setting, computing this power of two requires only one more matrix multiplication since

$$(I - PS)^2 = I - 2PS + PSPS \tag{93}$$

$$= I - PS \cdot (2I - PS). \tag{94}$$

So, $P' = PS \cdot (2I - PS)$ can be used as a new - and certainly better - starting point for the Neumann algorithm. The iteration steps would then write as

$$x_k = (I - P')^k g \tag{95}$$

which clearly converges faster with a numerical effort that equals the one of a single iteration, $\mathcal{O}(n^2)$.

8.3. Approximate Duals on Quasi-Regular Lattices

For regular lattices, the approximation of a dual Gabor frame via preconditioning requires the computation of a single atom which is then shifted in time and frequency to build the approximate dual Gabor system. This is possible due to the underlying group structure. In a non-regular situation, this is not possible anymore. Even in the analytic case, a dual frame is in general, not a Gabor frame, i.e. a collection of TF-shifts of a single atom $\tilde{g}$. Instead, one would need to solve the system of linear equations $\mathbf{S}_{g,\Lambda}\tilde{g} = g$ directly, which is numerically very expensive. We can use the idea of preconditioning also in this setting to approximate dual atoms locally for every lattice point and build a global approximate dual system with them. This idea has been implemented for *non-stationary Gabor frames* [3], where the

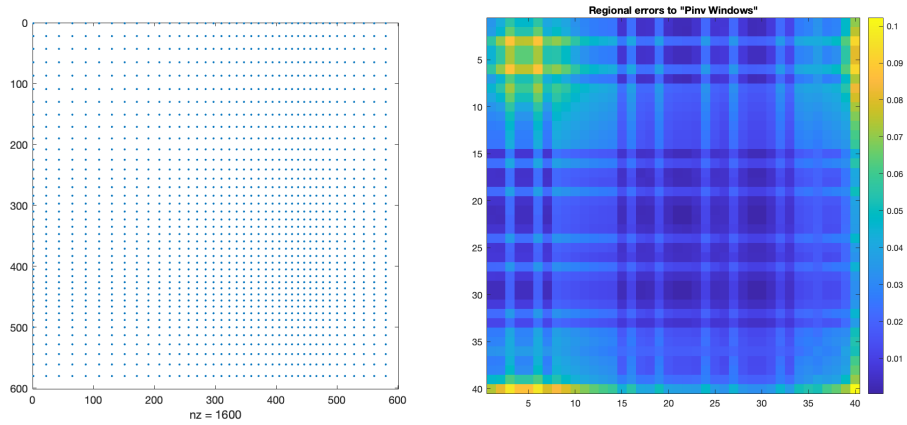

Figure 6: Left: Lattice with parameters slowly varying from 10 to 22 for a signal of length 600. This yields a maximum redundancy in the resulting Gabor system of 6 and a minimum of 1.24. Right: Distances of the preconditioned and TF-shifted duals to the duals coming from the pseudo-inverse of the Gabor matrix.

authors use single preconditioning locally to obtain an approximate global dual, see [19].

Here we would like to emphasize the case where the lattice is *quasi-regular*, i.e. the lattice parameters a, b are slowly varying in time and frequency simultaneously. The left plot in Figure 6 shows an example of such a quasi-regular lattice. By the continuity results in Section 5 we can expect the double preconditioning method to work reasonably well in this setting. An approximate dual can be found by using local lattice parameters that are derived individually for every lattice point. Furthermore, the analytic construction of the double preconditioner derived in Section 3.1 makes it possible to choose the local parameters freely without the divisibility necessity of $\frac{n}{a}, \frac{n}{b}$ to be integers. In the presented experimental setup we computed the local lattice parameter to be the rounded averages of the distances to the neighboring points.

The right plot in Figure 6 shows the regional errors of the preconditioned approximated duals to the dual system computed with the pseudo-inverse of the Gabor matrix, consisting of the time-frequency-shifted versions of g according to the quasi-regular lattice. One clearly recognizes the region where the lattice is densest from the error values. The slices occur at the quanti-

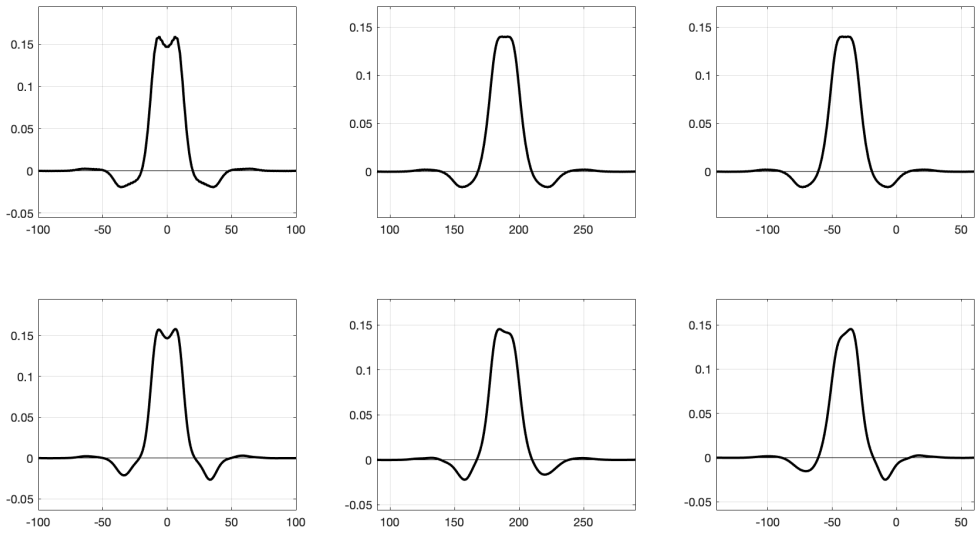

Figure 7: Upper row: A selection of approximate dual windows via double preconditioning. Lower row: Exact dual windows computed using the pseudo-inverse.

zation steps, i.e. where the rounded averages jump to the next integer and thus, produce a larger error.

We note that the choice of the local parameters (e.g. by using local averages as in our case) is somewhat arbitrary and requires some quantization in the discrete case. However, in the discussion in Section 5 on the continuous dependence we provided arguments why it is actually not crucial *how* quantization is done in particular.

Figure 7 shows a direct comparison between the windows for three typical positions within the quasi-regular family. Notice that the preconditioned windows (upper row) have symmetric shapes as they arise from regular Gabor systems. In contrast, the windows arising in the canonical dual of the system (lower row) have to take into account the varying lattice parameters, they are asymmetric. Clearly, the preconditioned windows do not "know" their neighbors. The resulting approximate dual system yields a reconstruction error that is at 10^{-10} after 10 iterations of the above iteration scheme. In fact, this approach provides good approximate dual Gabor atoms at each point and it seems that under suitable conditions (to be discussed in subsequent work) the overall Gabor family, with now slowly varying atoms will constitute

an *approximate dual frame* in the spirit of [14].

9. Conclusion and Outlook

In this paper, we try to move the idea of double or even multiple preconditioning (i.e. concatenation of various preconditioners arising from different commutative subgroups of $\Lambda^\perp$) for Gabor families from a purely experimental observation as presented in [4] to a more detailed mathematical analysis, by discussing algebraic, numerical, and functional analytic aspects. The starting point is the Janssen representation of a Gabor frame operator $S = S_{g,\Lambda}$, which is valid for general lattice Λ, and describes the spreading representation of of S, which is supported on the adjoint lattice. For the classical case of Gabor families arising from (g, a, b) this is equivalent to the Walnut representation.

The interpretation of the two preconditioners as the inverse to best approximations (in a suitable sense) of S in the spreading domain, and the fact that they are making use of commutative (in the TF-sense) subgroups of Λ° ensures that they are easy to realize and also explains, among others, why and when they commute or almost commute.

Finally, we indicate the usefulness of the approach to quasi-regular situations, where one has to compute a large number of (approximate) dual atoms. For reconstruction tasks, such approximate dual Gabor families can be employed to obtain perfect reconstruction from samples of the STFT by running a few iterations.

The expression in (28) can be easily extended for any cyclic subgroup of Λ°, different from the horizontal or vertical axis in the phase space description of the Gabor frame operator (via the Janssen representation). In this way, one can define alternative preconditioners based on similar principles. The choice and the order of such preconditioners, which might be more natural e.g. in the context of a hexagonal lattice, remains the subject of further investigations. We also see certain connections to the well-known POCS method (Projection Onto Convex Sets), here with affine subspaces arising from the Wexler-Raz relation, which also includes the question in which order such a sequence of projections will show optimal performance.

It will be interesting to further investigate the possible benefits of the double preconditioning method for cases (perhaps multidimensional, non-separable ones) where direct inversion is not possible, and where it is impor-

tant to improve the range of sufficient conditions (combined with effective estimates for the frame bounds) for Gabor frames.

Another direction that may be fruitful in the future involves weighted versions of this Banach algebra used so far, namely $\mathcal{A}_w^{\Lambda^\circ}$, where w is some *submultiplicative* (Beurling) weight function on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, sampled at the lattice points Λ° (hence it will be a submultiplicative weight on Λ° as a group of its own right), with then (finiteness of the following norm providing the definition)

$$\|T \,|\, \mathcal{A}_w^{\Lambda^\circ}\| = \sum_{\lambda^\circ \in \Lambda^\circ} |c_{\lambda^\circ}| w(\lambda) < \infty. \tag{96}$$

With this choice, the space $\left(\boldsymbol{\ell}_w^1, \|\cdot\|_{\boldsymbol{\ell}_w^1}\right)$ is again a solid Banach algebra with respect to convolution (see [58] for concerning *Beurling algebras*). Thus arguments similar to the unweighted case can be applied.

We do not go into a detailed discussion of the Janssen criterion for this case, but it is clear that stronger weights will make it more difficult to satisfy the corresponding estimate for $\gamma < 1$. In some cases, one may have to elaborate on the tradeoff between the closeness of the corresponding value $\gamma = \gamma(w)$ to one (hence a big factor of the form $1/(1-\gamma)$ in the estimate of the inverse) and the stronger decay enforced by the stronger weight, as a property of the concentration of the inverse operator near the identity operator.

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APPENDIX 1: Best Approximation by Circulant Matrices

We derive that the best approximation of $\mathbf{S}_{g,\Lambda}$ arises by taking the means of all side diagonals. Any $n \times n$ matrix A has the Fourier representation

$$A^{\mathcal{F}}[k,l] = \frac{1}{n} \sum_{r,p=1}^{n} A[r,p] e^{\frac{-2\pi i(kr-lp)}{n}}.$$

The best approximation of $A^{\mathcal{F}}$ can be written (using MATLAB notation) as

$$C_A^{\mathcal{F}} = \mathtt{diag}\left(\mathtt{diag}\left(A^{\mathcal{F}}\right)\right).$$

Back on the time side, this writes as

$$C_A[s,t] = \frac{1}{n} \sum_{k,l=1}^{n} \left(A^{\mathcal{F}}[k,l]\delta_{k-l}\right) e^{\frac{2\pi i(ks-lt)}{n}}.$$

Via a change of indices, we find a representation that emphasizes the structure of the side-diagonals,

$$\begin{aligned} C_A[s,s-u] &= \frac{1}{n} \sum_{k=1}^{n} \widehat{A}[k,k] e^{\frac{2\pi iku}{n}} \\ &= \frac{1}{n} \sum_{k=1}^{n} \frac{1}{n} \sum_{r,p=1}^{n} A[r,p] e^{\frac{-2\pi ik(r-p)}{n}} e^{\frac{2\pi iku}{n}} \\ &= \frac{1}{n^2} \sum_{r,q=1}^{n} A[r,r-q] \sum_{k=1}^{n} e^{\frac{-2\pi ik(q-u)}{n}} \\ &= \frac{1}{n} \sum_{r=1}^{n} A[r,r-u]. \end{aligned}$$

So, the best approximation of a matrix (in the Frobenius norm sense) by a circulant matrix is the convolution matrix where one takes the mean of each side-diagonal. In fact, this is quite natural as a circulant matrix is one where the side diagonals are constant.

APPENDIX 2

Lemma: *Let T_α, $\alpha \in I$ be a bounded net of linear operators on a Banach spaces $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, such that for any f from a dense subset D one has:*

$$\lim_{\alpha\to\infty} T_\alpha(f) = T_0(f), \quad \textit{in } (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}).$$

Then one has uniform convergence of this net over any compact subset $M \subset \boldsymbol{B}$, i.e. given $\varepsilon > 0$ there exists α_0 such that for $\alpha \succeq \alpha_0$ one has

$$\|T_\alpha(f) - T_0(f)\|_{\boldsymbol{B}} \leq \varepsilon, \quad f \in M.$$

Proof. The proof is more or less an elementary exercise in functional analysis. Let us assume that we have $\sup_{\alpha \in I} |||T|||_{\boldsymbol{B}\to\boldsymbol{B}} = C_0 < \infty$.

By the compactness of M we find a finite set $F = \{f_1, ..., f_K\} \subset M$ such that we can find for any given $f \in M$ some $f_k \in F$ with $\|f - f_k\|_{\boldsymbol{B}} < \varepsilon/(4C_0)$.

The concept of a convergent net guarantees that for $\varepsilon > 0$ there exists $\alpha_0 \in I$ such that $\alpha \succeq \alpha_0$ implies

$$\|T_\alpha(f_k) - T_0(f_k)\|_{\boldsymbol{B}} \leq \varepsilon/2, \quad k = 1, ..., K.$$

Together these estimates yield for a given $f \in M$, by choosing k in a suitable way:

$$\|T_\alpha(f) - T_0(f)\|_{\boldsymbol{B}} \leq \|T_\alpha(f - f_k)\|_{\boldsymbol{B}} + \|T_\alpha(f_k) - T_0(f_k)\|_{\boldsymbol{B}} + \|T_0(f - f_k)\|_{\boldsymbol{B}}. \tag{.1}$$

Since we have for $h \in D$:

$$\|T_0(h)\|_{\boldsymbol{B}} = \lim_{\alpha\to\infty} T_\alpha(h) \leq C_0 \|h\|_{\boldsymbol{B}}$$

the mapping $h \mapsto T_0(h)$ is obviously a linear and extends uniquely to a bounded linear mapping with $|||T_0|||_{\boldsymbol{B}\to\boldsymbol{B}} \leq C_0$. For the last term, we have

$$\|T_0(f - f_k)\|_{\boldsymbol{B}} \leq C_0 \|f - f_k\|_{\boldsymbol{B}},$$

the same estimate as the first term, so altogether we obtain for any $\alpha \succeq \alpha_0$:

$$\|T_\alpha(f) - T_0(f)\|_{\boldsymbol{B}} \leq 2\, C_0\, \|f - f_k\|_{\boldsymbol{B}} + \varepsilon/2 < \varepsilon,$$

as was claimed. □

References

[1] S. Awodey. *Category Theory 2nd ed.* Oxford Logic Guides 52. Oxford: Oxford Science Publications. Oxford: Oxford University Press. xv, 311 p., 2010.

[2] P. Balazs. Basic definition and properties of Bessel multipliers. *J. Math. Anal. Appl.*, 325(1):571–585, 2007.

[3] P. Balazs, M. Dörfler, F. Jaillet, N. Holighaus, and G. A. Velasco. Theory, implementation and applications of nonstationary Gabor frames. *J. Comput. Appl. Math.*, 236(6):1481–1496, 2011.

[4] P. Balazs, H. G. Feichtinger, M. Hampejs, and G. Kracher. Double preconditioning for Gabor frames. *IEEE Trans. Signal Process.*, 54(12):4597–4610, December 2006.

[5] P. Balazs and K. Gröchenig. A guide to localized frames and applications to Galerkin-like representations of operators. In I. Pesenson, H. Mhaskar, A. Mayeli, and et.al., editors, *Novel Methods in Harmonic Analysis with Applications to Numerical Analysis and Data Processing*, Appl. Numer. Harm. Anal.,, pages 47–79. Birkhäuser/Springer, 2017.

[6] S. Bannert. Banach-Gelfand Triples and Applications in Time-Frequency Analysis. Master's thesis, University of Vienna, 2010.

[7] H. Bölcskei and A. J. E. M. Janssen. Equivalence of two methods for constructing tight Gabor frames. *IEEE Signal Processing Letters*, 7(4):79–82, April 2000.

[8] P. G. Casazza. The art of frame theory. *Taiwanese J. Math.*, 4(2):129–201, 2000.

[9] P. G. Casazza and O. Christensen. Weyl-Heisenberg frames for subspaces of $L^2(R)$. *Proc. Amer. Math. Soc.*, 129(1):145–154 (electronic), 2001.

[10] R. Chan and G. Strang. Toeplitz equations by conjugate gradients with circulant preconditioner. *SIAM J. Sci. Statist. Comput.*, 10(1):104–119, 1989.

[11] O. Christensen. Atomic decomposition via projective group representations. *Rocky Mountain J. Math.*, 26(4):1289–1312, 1996.

[12] O. Christensen. Moment problems and stability sesults for frames with applications to irregular sampling and Gabor frames. *Appl. Comput. Harmon. Anal.*, 3(1):82–86, 1996.

[13] O. Christensen. *An Introduction to Frames and Riesz Bases*. Applied and Numerical Harmonic Analysis. Birkhäuser Basel, Second edition, 2016.

[14] O. Christensen and R. S. Laugesen. Approximately dual frames in Hilbert spaces and applications to Gabor frames. *Sampl. Theory Signal Image Process.*, 9(1-3):77–89, 2010.

[15] E. Cordero, H. G. Feichtinger, and F. Luef. Banach Gelfand triples for Gabor analysis. In *Pseudo-differential Operators*, volume 1949 of *Lect. Notes Math.*, pages 1–33. Springer, Berlin, 2008.

[16] I. Daubechies. The wavelet transform, time-frequency localization and signal analysis. *IEEE Trans. Inform. Theory*, 36(5):961–1005, 1990.

[17] I. Daubechies, A. Grossmann, and Y. Meyer. Painless nonorthogonal expansions. *J. Math. Phys.*, 27(5):1271–1283, May 1986.

[18] F. De Mari, H. G. Feichtinger, and K. Nowak. Uniform eigenvalue estimates for time-frequency localization operators. *J. London Math. Soc.*, 65(3):720–732, 2002.

[19] M. Dörfler and E. Matusiak. Nonstationary Gabor frames - approximately dual frames and reconstruction errors. *Adv. Comput. Math.*, 41(2):293–316, April 2015.

[20] Y. C. Eldar and G. D. j. Forney. Optimal tight frames and quantum measurement. *IEEE Trans. Inform. Theory*, 48(3):599–610, 2002.

[21] Y. C. Eldar and T. Werther. General framework for consistent sampling in Hilbert spaces. *Int. J. Wavelets Multiresolut. Inf. Process.*, 3(3):347–359, 2005.

[22] H. Feichtinger, S. Pilipovic, and B. Prangoski. Modulation spaces associated to tensor products of amalgam spaces. *Annali Pura Appl. Math.*, pages 127–155, 2022.

[23] H. G. Feichtinger. Multipliers from $L^1(G)$ to a homogeneous Banach space. *J. Math. Anal. Appl.*, 61:341–356, 1977.

[24] H. G. Feichtinger. On a new Segal algebra. *Monatsh. Math.*, 92:269–289, 1981.

[25] H. G. Feichtinger. A compactness criterion for translation invariant Banach spaces of functions. *Analysis Mathematica*, 8:165–172, 1982.

[26] H. G. Feichtinger. Banach convolution algebras of Wiener type. In *Proc. Conf. on Functions, Series, Operators, Budapest 1980*, volume 35 of *Colloq. Math. Soc. Janos Bolyai*, pages 509–524. North-Holland, Amsterdam, Eds. B. Sz.-Nagy and J. Szabados. edition, 1983.

[27] H. G. Feichtinger. Modulation spaces on locally compact Abelian groups. Technical report, University of Vienna, January 1983.

[28] H. G. Feichtinger. Compactness in translation invariant Banach spaces of distributions and compact multipliers. *J. Math. Anal. Appl.*, 102:289–327, 1984.

[29] H. G. Feichtinger. Spline-type spaces in Gabor analysis. In D. X. Zhou, editor, *Wavelet Analysis: Twenty Years Developments Proceedings of the International Conference of Computational Harmonic Analysis, Hong Kong, China, June 4–8, 2001*, volume 1 of *Ser. Anal.*, pages 100–122. World Sci.Pub., River Edge, NJ, 2002.

[30] H. G. Feichtinger. Homogeneous Banach spaces as Banach convolution modules over $M(G)$. *Mathematics*, 10(3):1–22, 2022.

[31] H. G. Feichtinger and P. Gröbner. Banach spaces of distributions defined by decomposition methods. I. *Math. Nachr.*, 123:97–120, 1985.

[32] H. G. Feichtinger and K. Gröchenig. Gabor frames and time-frequency analysis of distributions. *J. Funct. Anal.*, 146(2):464–495, 1997.

[33] H. G. Feichtinger and N. Kaiblinger. Varying the time-frequency lattice of Gabor frames. *Trans. Amer. Math. Soc.*, 356(5):2001–2023, 2004.

[34] H. G. Feichtinger and N. Kaiblinger. Quasi-interpolation in the Fourier algebra. *J. Approx. Theory*, 144(1):103–118, 2007.

[35] H. G. Feichtinger and W. Kozek. Quantization of TF lattice-invariant operators on elementary LCA groups. In H. G. Feichtinger and T. Strohmer, editors, *Gabor analysis and algorithms*, Appl. Numer. Harmon. Anal., pages 233–266. Birkhäuser, Boston, MA, 1998.

[36] H. G. Feichtinger, W. Kozek, and F. Luef. Gabor Analysis over finite Abelian groups. *Appl. Comput. Harmon. Anal.*, 26(2):230–248, 2009.

[37] H. G. Feichtinger and F. Luef. Wiener amalgam spaces for the Fundamental Identity of Gabor Analysis. *Collect. Math.*, 57(Extra Volume (2006)):233–253, 2006.

[38] H. G. Feichtinger, F. Luef, and T. Werther. A guided tour from linear algebra to the foundations of Gabor analysis. In *Gabor and Wavelet Frames*, volume 10 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 1–49. World Sci. Publ., Hackensack, 2007.

[39] H. G. Feichtinger and K. Nowak. A first survey of Gabor multipliers. In H. G. Feichtinger and T. Strohmer, editors, *Advances in Gabor Analysis*, Appl. Numer. Harmon. Anal., pages 99–128. Birkhäuser, 2003.

[40] H. G. Feichtinger and T. Strohmer. *Gabor Analysis and Algorithms. Theory and Applications.* Birkhäuser, Boston, 1998.

[41] H. G. Feichtinger and G. Zimmermann. A Banach space of test functions for Gabor analysis. In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Applied and Numerical Harmonic Analysis, pages 123–170. Birkhäuser Boston, 1998.

[42] M. Fornasier and K. Gröchenig. Intrinsic localization of frames. *Constr. Approx.*, 22(3):395–415, 2005.

[43] K. Gröchenig. *Foundations of Time-Frequency Analysis.* Appl. Numer. Harmon. Anal. Birkhäuser, Boston, MA, 2001.

[44] K. Gröchenig and M. Leinert. Wiener's lemma for twisted convolution and Gabor frames. *J. Amer. Math. Soc.*, 17:1–18, 2004.

[45] K. Gröchenig and J. Stöckler. Gabor frames and totally positive functions. *Duke Math. J.*, 162(6):1003–1031, 2013.

[46] C. Heil. An introduction to weighted Wiener amalgams. In M. Krishna, R. Radha, and S. Thangavelu, editors, *Wavelets and their Applications (Chennai, January 2002)*, pages 183–216. Allied Publishers, New Delhi, 2003.

[47] M. Jakobsen and F. Luef. Sampling and periodization of generators of Heisenberg modules. *International Journal of Mathematics*, 30(10):1950051, 2019.

[48] M. S. Jakobsen. On a (no longer) New Segal Algebra: a review of the Feichtinger algebra. *J. Fourier Anal. Appl.*, 24(6):1579–1660, 2018.

[49] T. Kloos, J. Stöckler, and K. Gröchenig. Implementation of discretized Gabor frames and their duals. *IEEE Trans. Inform. Theory*, 62(5):2759–2771, 2016.

[50] S. MacLane. *Categories for the working mathematician. 4th corrected printing.* Graduate Texts in Mathematics, 5. New York etc.: Springer-Verlag. ix, 1988.

[51] I. Oseledets and E. Tyrtyshnikov. A unifying approach to the construction of circulant preconditioners. *Linear Algebra and its Applications*, 418(2-3):435–449, 2006.

[52] N. Perraudin, N. Holighaus, P. Sondergaard, and P. Balazs. Designing Gabor windows using convex optimization. *Appl. Math. Comput.*, 330:266–287, 2018.

[53] S. Qiu. Structural properties of the Gabor transform and numerical algorithms. *Proc. SPIE, vol. 2491, Wavelet Applications for Dual Use*, pages 980–991, 1995.

[54] S. Qiu. Generalized dual Gabor atoms and best approximations by Gabor family. *Signal Process.*, 49:167–186, 1996.

[55] S. Qiu and H. G. Feichtinger. The structure of the Gabor matrix and efficient numerical algorithms for discrete Gabor expansion. In *Proc. SPIE VCIP94, SPIE 2308*, pages 1146–1157, Chicago, 1994.

[56] S. Qiu and H. G. Feichtinger. Discrete Gabor structures and optimal representation. *IEEE Trans. Signal Process.*, 43(10):2258–2268, October 1995.

[57] H. Reiter. *Classical Harmonic Analysis and Locally Compact Groups*. Clarendon Press, Oxford, 1968.

[58] H. Reiter and J. D. Stegeman. *Classical Harmonic Analysis and Locally Compact Groups. 2nd ed.* Clarendon Press, Oxford, 2000.

[59] P. L. Sondergaard. Gabor frames by sampling and periodization. *Adv. Comput. Math.*, 27:355–373, 2005.

[60] T. Strohmer. Numerical algorithms for discrete Gabor expansions. In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Appl. Numer. Harmon. Anal., pages 267–294. Birkhäuser Boston, 1998.

[61] T. Tschurtschenthaler. The Gabor Frame Operator (its Structure and Numerical Consequences). Master's thesis, University of Vienna, 2000.

[62] D. F. Walnut. Continuity properties of the Gabor frame operator. *J. Math. Anal. Appl.*, 165(2):479–504, 1992.

[63] A. Weil. *L'integration dans les Groupes Topologiques et ses Applications.* Hermann and Cie, Paris, 1940.

[64] Y. Zhang. Approximate complementation and its applications in studying ideals of Banach algebras. *Math. Scand.*, 92(2):301–308, 2003.